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SEMI-SQUARE-SUMMABLE FOURIER-STIELTJES TRANSFORMS

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For G a locally compact abelian group with dual Γ, let μ be a (finite regular Borel) measure on G with Fourier-Stieltjes transform $\hat{\mu}$. Doss has recently shown that when Γ is (algebraically) a totally ordered abelian group and $\hat{\mu}$ is square integrable on the negative half Γ_- of Γ then its singular component σ has $\hat{\sigma} = 0$ on Γ_-; in particular $\mu E = 0$ for each common null set E of the analytic measures (those with transforms 0 on Γ_-), such E being Haar-null.

In the similar (but usually distinct) case in which Γ is partially ordered by a nonzero homomorphism $\psi: \Gamma \to R$ with $\Gamma_- = \psi^{-1}(-\infty, 0]$ the common null sets E are known, and our purpose is to note in this setting how function algebra results apply to show $\mu E = 0$ when $\hat{\mu} \in L^2(\Gamma_-)$, and when $\hat{\mu}$ satisfies sometimes weaker (but more obscure) hypotheses.

Doss' results appear in [2], and the function algebra results we apply are those in [4, § 1], [5, § 2], with which we shall assume the reader familiar. The common null sets mentioned above are given in [1, 5].

THEOREM 1. *Let ψ, Γ_- be as above and let $\varphi: R \to G$ be the homomorphism dual to ψ. If*

$$\int_{\Gamma_-} |\hat{\mu}(\gamma)|^2 \, d\gamma < \infty \tag{1}$$

then μ vanishes on all Borel $E \subset G$ for which

$$\{t \in R: x + \varphi(t) \in E\} \text{ has linear measure } 0, \tag{2}$$

for all $x \in G$, i.e. (by definition [3, § 2]) μ is absolutely continuous in the direction of φ.

Proof. Let G^a be the Bohr compactification of G, with dual Γ_d, the discrete version of Γ. Dual to $\psi: \Gamma_d \to R$ we have a map of R into G^a, the composition $R \xrightarrow{\varphi} G \to G^a$, which we still call φ. Note that each Borel E in G is Borel[1] in G^a, and if $E \subset G$ satisfies (2) for $x \in G$ it does for all x in G^a (the set is void for $x \in G^a \backslash G$). As in [5] we are forced to transfer our attention to G^a to apply the function algebra results.

[1] We take the σ-ring generated by compacta as our Borel sets.

Let A be the closed span of $\Gamma_+ = \psi^{-1}[0, \infty)$ in $C(G^a)$, a subalgebra of $C(G^a)$. As usual we can shift μ to a measure on G^a carried by its subset G [6] with the same Fourier-Stieltjes transform as before. Let $\hat{f}$ be the element $\hat{\mu}\chi$ of $L^2(\Gamma)$, where χ is the characteristic function of Γ_-, and[2] f the element of $L^2(G)$ corresponding to $\hat{f}$.

For any trigonometric polynomial $p = \sum c_i \gamma_i$ in A (i.e., with $\psi(\gamma_i) \geqq 0$) we have

$$(p\mu)^\wedge(\gamma) = \int \bar{\gamma} p d\mu = \sum c_i \hat{\mu}(\gamma - \gamma_i) = (\sum c_i \delta_{-\gamma_i}) * \hat{\mu}(\gamma) ,$$

and since

$$\begin{aligned}(\sum c_i \delta_{-\gamma_i}) * \hat{f}(\gamma) &= \sum c_i \hat{\mu}(\gamma - \gamma_i)\chi(\gamma - \gamma_i) \\ &= \sum c_i \hat{\mu}(\gamma - \gamma_i) = (p\mu)^\wedge(\gamma)\end{aligned}$$

if $\psi(\gamma) \leqq 0$, we have

$$\int_{\Gamma_-} |(p\mu)^\wedge|^2 \, d\gamma \leqq \int_\Gamma |(\sum c_i \delta_{-\gamma_i}) * \hat{f}|^2 \, d\gamma = \int_G |pf|^2 \, dx \leqq ||p||_\infty^2 \, ||f||_2^2 ,$$

or

$$||(p\mu)^\wedge \chi||_2 \leqq ||f||_2 \cdot ||p||_\infty \, . \tag{3}$$

Now (3) continues to hold for any $a \in A$ in place of $p \in A$: for if $p_n \to a$ in A then $(p_n\mu)^\wedge \to (a\mu)^\wedge$ uniformly, so that for any compact $K \subset \Gamma_-$

$$\begin{aligned}\int_K |(a\mu)^\wedge|^2 \, d\gamma &= \lim \int_K |(p_n\mu)^\wedge|^2 \, d\gamma \leqq ||f||_2^2 \lim ||p_n||_\infty^2 \\ &= ||f||_2^2 \, ||a||_\infty^2\end{aligned}$$

whence $||(a\mu)^\wedge\chi||_2 \leqq ||f||_2 \, ||a||_\infty$. Indeed this clearly follows whenever $||p_n||_\infty \leqq ||a||_\infty$ and $(p_n\mu)^\wedge \to (a\mu)^\wedge$ uniformly.

Let γ be a fixed element of Γ with $\psi(\gamma) > 0$, and let $\mu = \nu + \sigma$ be the Lebesgue decomposition of μ relative to M^γ (the probability measures on G^a orthogonal to γA, cf. [4, §1]), with $\nu \ll M^\gamma$, σ M^γ-singular. By the argument of the last paragraph of [5, §2], ν vanishes on Borel sets in G^a satisfying (2), so we can complete our proof by showing $\sigma = 0$. As in [4] σ is carried by $\bigcup K_n$, where K_n is a compact M^γ-null set.

By the abstract Forelli Lemma [4, 1.2] (applied to the algebra $\mathbf{C} + \gamma A$) and dominated convergence we have $\{a_n\}$ in the unit ball of A for which $a_n\mu \to \sigma$ in norm, so $(a_n a\mu)^\wedge \to (a\sigma)^\wedge$ uniformly and again we conclude that $||(a\sigma)^\wedge\chi||_2 \leqq ||f||_2 \, ||a||_\infty$ for $a \in A$.

[2] It should be noted that when G is compact $f \in L^1(G)$ and the result follows trivially from [**1**]; for then $\nu(dx) = \mu(dx) - f(x)dx$ defines an analytic measure.

Now by [5, §2] each measure τ on G^a orthogonal to A has $\tau_{K_n} = 0$ for each K_n and thus by [3, 4.8] K_n is an intersection of peak sets of A, and an interpolation set for A; using the regularity of σ one then concludes[3] there is a sequence $\{a_j\}$ in the unit ball of A for which $a_j\sigma \to |\sigma_{K_n}|$ in norm. So again $\| |\sigma_{K_n}|^\wedge \cdot \chi \|_2 \leqq \|f\|_2 \cdot 1$, which of course implies $|\sigma_{K_n}|^\wedge \in L^2(\Gamma)$ since the absolute value of this function is even. Because μ is carried by the subset G of G^a, the same is true of its restrictions σ and σ_{K_n} and so, as a measure on G with square summable transform, $|\sigma_{K_n}|$ is absolutely continuous by the elementary argument given by Doss [2, Th. 1]. Hence σ is absolutely continuous.

To complete our proof we can show $\sigma = 0$ by showing σ is carried by a Haar-null set. And since σ is carried by a σ-compact set, it suffices to show σ_{x_0+V} is carried by a Haar null set for each $x_0 \in G$ and some compact symmetric neighborhood V of the identity. But σ and each $\lambda \in M^\gamma$ are mutually singular, so it suffices to show there is a λ in M^γ equivalent to Haar measure on $x_0 + V$, and, for example, with m Haar measure

$$\lambda E = \int_{-\infty}^{\infty} \int_E \frac{1}{m2V} \chi_{x_0+2V}(x - \varphi(t)) dx \rho(t) dt$$

defines such a measure if

$$\hat{\rho}(s) = \begin{cases} 1 - \dfrac{|s|}{\psi(\gamma)}, & |s| \leqq \psi(\gamma) \\ 0 & \text{elsewhere} . \end{cases}$$

Indeed

$$\rho(t) = \psi(\gamma)\left(\frac{\sin t\psi(\gamma)/2}{t/2}\right)^2 \geqq 0$$

so $\lambda \geqq 0$ and

$$\hat{\lambda}(\gamma) = \frac{1}{m2V} \hat{\chi}_{x_0+2V}(\gamma) \cdot \hat{\rho}(\psi(\gamma)) ,$$

as is easily verified; so $\hat{\lambda}$ vanishes off $\psi^{-1}(-\psi(\gamma), \psi(\gamma))$ whence λ is orthogonal to γA, the span of $\{\beta \in \Gamma : \psi(\beta) \geqq \psi(\gamma)\}$. And $\lambda E = 0$ implies

$$\int_E \chi_{x_0+2V}(x - \varphi(t_0)) dx = 0$$

for some t_0 with $\varphi(t_0) \in V$ since $\rho(t) > 0$ a.e., $\varphi(0) \in V$ and φ is con-

[3] By regularity there is a peak set (an intersection of countably many such) $F_n \supset K_n$ for which $\sigma_{F_n} = \sigma_{K_n}$, and if f peaks on F_n then $f^k \to 1$ a.e. $|\sigma_{K_n}|$, $\to 0$ a.e. $|\sigma_{K'_n}|$. If $\sigma_{K_n} = \rho |\sigma_{K_n}|$, $|\rho| \equiv 1$, then we have f_k in the unit ball of $C(K_n)$ for which $f_k \to \rho$ a.e. $|\sigma_{K_n}|$, hence b_k in the $(1+\varepsilon)$-ball of A for which $b_k = f_k$ on K_n, whence $b_k f^k \sigma \to |\sigma_{K_n}|$ by dominated convergence.

tinuous, so if $E \subset x_0 + V$ we have $E - \varphi(t_0) \subset x_0 + 2V$, and therefore

$$0 = \int_E \chi_{x_0+2V}(x - \varphi(t_0))dx = \int_E 1dx = mE\ .$$

Hence $m_{x_0+2V} \ll \lambda_{x_0+2V}$; the reverse is obvious (and actually unnecessary) and our proof complete.

Variants of theorem 1 can be obtained from the same argument, but seem to require more artificial hypotheses. For example

THEOREM 2. *With ψ, Γ_- as before, suppose the continuous function $f = f^* \in L^1(G) \cap L^1(\Gamma)^\wedge$ never vanishes on*[4] *G, and μ is a measure for which for some k*

$$(4) \qquad \int_\Gamma \left| \int_{\Gamma_-} (p\mu)^\wedge(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta \leqq k\,||p||_\infty$$

for all trigonometric polynomials $p = \sum_{i=1}^n c_i\gamma_i$ with $\psi(\gamma_i) \geqq 0$. Then $\mu E = 0$ for each Borel $E \subset G$ satisfing (2).

We argue exactly as before that if $p_n\mu \to a\mu$ and $||p_n||_\infty \leqq ||a||_\infty$, $a \in A$, one has

$$\int_K \left| \int_{\Gamma_-} (a\mu)^\wedge(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta \leqq k\,||a||_\infty$$

for K compact, so (4) holds for p an arbitrary element of A.

With $\mu = \nu + \sigma$ as before we again obtain (4) for $p \in A$ and σ in place of μ, and then for $1 = p \in A$ and $|\sigma_{K_n}| = \tau$ in place of σ. But since $\bar{\hat{\tau}}(-\gamma) = \hat{\tau}(\gamma)$ the finite integral

$$(5) \qquad \int_\Gamma \left| \int_{\Gamma_-} \hat{\tau}(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta$$

coincides with

$$\begin{aligned}(6) \qquad \int_\Gamma \left| \int_{\Gamma_-} \bar{\hat{\tau}}(-\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta &= \int_\Gamma \left| \int_{\Gamma_-} \bar{\hat{\tau}}(-\gamma)\bar{\hat{f}}(\gamma - \beta)d\gamma \right|^2 d\beta \\ &= \int_\Gamma \left| \int_{\Gamma_+} \hat{\tau}(\gamma)\hat{f}(-\gamma - \beta)d\gamma \right|^2 d\beta \\ &= \int_\Gamma \left| \int_{\Gamma_+} \hat{\tau}(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta\end{aligned}$$

so that, by Minkowski, $\hat{\tau} * \hat{f} \in L^2(\Gamma)$. Trivially one verifies that the transform of the finite measure $f\tau$ on G is $\hat{\tau} * \hat{f}$: thus $f\tau$ is absolutely

[4] When such an f exists this contains the preceding result. For when $\hat{\mu}\chi \in L^2(\Gamma)$ so is $(p\mu)^\wedge\chi$ and always of norm $\leqq k\,||p||_\infty$ as we saw in the proof of Theorem 1. But then $||(p\mu)^\wedge\chi * \hat{f}||_2 \leqq ||(p\mu)^\wedge\chi||_2\,||\hat{f}||_1 \leqq k\,||p||_\infty\,||\hat{f}||_1$ which is (4).

continuous, so $\tau = |\sigma_{K_n}|$ is since f never vanishes; again σ is singular with respect to Haar measure, and $\sigma = 0$ follows.

THEOREM 3. *Suppose there are $\gamma_n \in \Gamma$ for which $\varepsilon_n = ||\bar{\gamma}_n\mu||_{A^*} \to 0$, where the norm is that of $\bar{\gamma}_n\mu$ as a functional on $A = \operatorname{span}\Gamma_+$. Then $\mu E = 0$ for every Borel E in G satisfying* (2).

We are supposing that $|(a\mu)^\wedge(\gamma_n)| \leqq \varepsilon_n ||a||_\infty$ for each $a \in A$, where $\varepsilon_n \to 0$. As before we have $a_j \in A$, $||a_j|| \leqq 1$, with $a_j\mu \to \sigma$, where σ is the M^γ-singular component of μ, so

$$|(a\sigma)^\wedge(\gamma_n)| \leqq \varepsilon_n ||a||_\infty \tag{7}$$

follows since $(a_j \cdot a\mu)^\wedge \to (a\sigma)^\wedge$ uniformly. Now we have σ carried by $\cup K_j$, K_j a compact M^γ-null set, and as before an intersection of peak sets of A and an interpolation set for A. So exactly as before (cf. footnote 3) we have $\{a_k\}$ in the unit ball of A for which $a_k\sigma \to \gamma_n |\sigma_{K_j}|$, whence by (7)

$$|(\gamma_n |\sigma_{K_j}|)^\wedge(\gamma_n)| = |\sigma_{K_j}|(1) = ||\sigma_{K_j}|| \leqq \varepsilon_n$$

for all n, so $\sigma_{K_j} = 0$, $\sigma = 0$, completing our proof as before.

As a final remark, we note that for any measure μ vanishing on all E satisfying (2), i.e., *for μ absolutely continuous in the direction of φ, if $|\psi(\gamma_n)| \to \infty$, we (at least) have $\bar{\gamma}_n\mu \to 0$ weakly.*[5] Indeed since $\Gamma\mu = \{\gamma\mu: \gamma \in \Gamma\}$ is conditionally weakly compact we need only see any weak cluster point of $\{\bar{\gamma}_n\mu\}$ must be 0, so it suffices to show

$$(\bar{\gamma}_n\mu)^\wedge(\gamma) = \hat{\mu}(\gamma + \gamma_n) \to 0 .$$

But this follows directly from the following easy "Riemann-Lebesgue lemma": *If μ is absolutely continuous in the direction of φ then for any $\varepsilon > 0$ there is an N for which $|\hat{\mu}(\gamma)| < \varepsilon$ if $|\psi(\gamma)| > N$.*

By [3, 2.4] μ translates continuously in the direction of φ, i.e., $||\mu - \mu_t|| < \varepsilon$ if $|t| < \delta$, where $\mu_t E = \mu(\varphi(t) + E)$. Thus for an appropriate continuous f on R vanishing off $(-\delta, \delta)$ we have

$$||\mu * f - \mu|| < \varepsilon ,$$

where

$$\mu * f = \int \mu_t f(t) dt$$

[5] Thus for any measure μ on G one has an analogue of a well known lemma of Helson: if $|\psi(\gamma_n)| \to \infty$, any weak cluster point ν of $\{\gamma_n\mu\}$ is carried by a subset E of G satisfying (2), i.e., null in the direction of φ in the terminology of [3]. (For ν is necessarily a weak cluster point of $\{\gamma_n\sigma\}$, where σ is the M^γ-singular component of μ, as always.)

can be interpreted as, say, a Riemann integral. But

$$
\begin{aligned}
(\mu * f)^\wedge(\gamma) &= \iint \overline{(x, \gamma)} \mu_t(dx) f(t) dt \\
&= \iint \overline{(x - \varphi(t), \gamma)} \mu(dx) f(t) dt \\
&= \hat{\mu}(\gamma) \int (\varphi(t), \gamma) f(t) dt \\
&= \hat{\mu}(\gamma) \int (t, \psi(\gamma)) f(t) dt = \hat{\mu}(\gamma) \hat{f}(-\psi(\gamma))
\end{aligned}
$$

which shows $(\mu * f)^\wedge$ has the desired property by the Riemann-Lebesgue lemma applied to f. As a uniform limit of such functions $\hat{\mu}$ of course has the same property.

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