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ON DYADIC SUBSPACES

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We prove a necessary condition for a (compact, Hausdorff) space to be dyadic (= image of product of 2-point spaces):

THEOREM. **Let Y be a dyadic space of weight m, and let r be a cardinal number less than m. Then X has a dyadic subspace of weight r.**

It may be observed (with the aid of Corollary 2, below) that this theorem is a stronger and more general version of a result published in a previous paper by the author [this Journal, 28 (1969), 173-182; Lemma III.6.]

A *dyadic space* is a Hausdorff space which is a continuous image of $\{0, 1\}^I$ (with the product topology) for some set I. Šanin has shown (see [2], Th. 1) that, if X is an infinite dyadic space, then the smallest possible cardinality for the exponent I is the weight of X, i.e., the least cardinality for a basis for the topology of X, hereinafter denoted by $w(X)$. Other observations concerning the significance of $w(X)$ for an infinite dyadic space include the following: Esenin-Volpin showed (see [3], Th. 4) that $w(X)$ is the least upper bound of the characters of the points of X; in [6] (Th. III.3) it is shown that a dyadic space having a dense subset of cardinality m must have weight no greater than 2^m. (The converse of this last statement follows from the well-known theorem of Hewitt, *et. al.*, in [4]).

In what follows we shall use, whenever necessary, the fact that, if X and Y are compact Hausdorff spaces and X is a continuous image of Y, then $w(X) \leqq w(Y)$. ([1], Appendix.) For a set S, $|S|$ denotes the cardinality of S.

2. **Proof of the theorem.** (1) Suppose X is a dyadic space and f a continuous function from $\{0, 1\}^I$ onto X. Define $\iota \in I$ to be *redundant* if, whenever two points p and q in $\{0, 1\}^I$ differ only in the ι *th* coordinate, we have $f(p) = f(q)$. By induction, if p and q differ only on a finite set of redundant coordinates, then $f(p) = f(q)$. Since f is continuous, we must have that $f(p) = f(q)$ whenever p and q differ only on an arbitrary set of redundant coordinates. Thus we may assume that all the indices in I are nonredundant.

(2) Given $\iota \in I$, there must exist two points $p = p^\iota$ and $q = q^\iota$ such that $p_\mu = q_\mu$ for all $\mu \neq \iota$, $p_\mu = 0$ for all but finitely many μ, and $f(p) \neq f(q)$; this follows from the continuity of f and the assumption that ι is nonredundant.

(3) Now let $r < w(X)$; if r is finite the conclusion is obvious.

Assuming r is infinite, choose a subset R_1 of I such that $|R_1| = r$. For each $\iota \in R_1$, choose p^ι and q^ι as in (2). Let

$$E_\iota = \{\mu \in R_1 : p^\iota_\mu = 1\} \cup \{\iota\}, \quad \text{and} \quad R = \bigcup \{E_\iota : \iota \in R_1\}.$$

Let $X_R = f(P_R)$, where $P_R = \{0,1\}^R \times \{0\} = \{p \in \{0,1\}^I : p_\mu = 0 \text{ for } \mu \notin R\}$. It is clear that $\{p^\iota : \iota \in R\} \cup \{q^\iota : \iota \in R\} \subset P_R$, and that $|R| = r$, so that $w(X_R) \leqq r$. We wish to show that $w(X_R) = r$; suppose $w(X_R) < r$, and let $\boldsymbol{B}$ be a basis for the topology of X_R with $|\boldsymbol{B}| = w(X_R)$. For each $\iota \in R_1$ there exist U and V, members of $\boldsymbol{B}$ with disjoint closures, such that $f(p^\iota) \in U$ and $f(q^\iota) \in V$. Since $r = |R_1| > |\boldsymbol{B} \times \boldsymbol{B}|$, there must exist U and V such that $R_2 = \{\iota : f(p^\iota) \in U, f(q^\iota) \in V\}$ has cardinality $> w(X_R)$. The choice function $\iota \to (p^\iota, q^\iota)$ is one-to-one, thus $\{(p^\iota, q^\iota) : \iota \in R_2\}$ has cardinality $> w(X_R)$, and we may as well assume that $\{p^\iota : \iota \in R_2\}$ is infinite. Since P_R is compact, there is an infinite net $\{p^\iota\}$ which converges to some point p^0, and since each p^ι differs from the corresponding q^ι only in a single coordinate, we must have that $\{q^\iota\}$ converges to p^0 also. But then $f(p^0) \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$, which we have assumed to be impossible. Thus $w(X_R) = r$. [*Note*: by a slight modification of the argument in this paragraph, we could take $R = I$ (containing only nonredundant indices) and get $|I| = w(X)$, as in Šanin's theorem.]

COROLLARY 1. *Every infinite dyadic space contains an infinite compact metric space.*

COROLLARY 2. *Every nonmetrizable dyadic space has a dyadic subspace of weight* $\aleph_1$.

COROLLARY 3. *Let X be a dyadic space, $w(X) = m$. Then X contains a chain $\{X_n : n \leqq m\}$ of dyadic subspaces with $w(X_n) = n$ for each $n \leqq m$.*

Proof. It is easy to see, in part (3) of the proof of the theorem, that if $w(X_R) = r < n$, we can choose $R' \supset R$ so that $w(X_{R'}) = n$. Clearly $X_{R'} \supset X_R$ if $R' \supset R$.

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REFERENCES

1. P. Alexandroff, *On bicompact extensions of topological spaces*, Mat. Sb. **5** (1939), 403-423 (Russian).
2. B. Efimov and R. Engelking, *Remarks on dyadic spaces II*, Colloq. Math. **13** (1965), 181-197.
3. R. Engelking and A. Pelczynski, *Remarks on dyadic spaces*, Colloq. Math. **11** (1963), 55-63.
4. E. Hewitt, *A remark on density characters*, Bull. Amer. Math. Soc. **52** (1946), 641-643.
5. J. L. Kelley, *General topology*, Van Nostrand, Princeton, 1955.
6. H. L. Peterson, *Regular and irregular measures on groups and dyadic spaces*, Pacific J. Math. **28** (1969), 173-182.

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