PACIFIC JOURNAL OF MATHFMATICS
Vol. 31, No. 3, 1969

SEQUENCES OF CONTRACTIVE MAPS AND FIXED POINTS

R. B. FRASER, JR., AND S. B. NADLER, JR.

Dedicated to Apollo XI

Let (X, ρ_0) be a metric space and, for each $n = 0, 1, 2, \cdots$, let $f_n: X \rightarrow X$ be a function with fixed point a_n. Assume that each function f_n is contractive with respect to a (possibly) different metric ρ_n, where each ρ_n is equivalent to ρ_0. This paper is concerned with the behavior of the sequence $\{a_n\}_{n=1}^{\infty}$ when $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f_0.

In §1 an example of a compact space (X, ρ_0) is given such that, though $\{\rho_n\}_{n=1}^{\infty}$ converges pointwise to ρ_0, $\{a_n\}_{n=1}^{\infty}$ converges, and f_n has ρ_n-Lipschitz constant $1/2$, $\{a_n\}_{n=1}^{\infty}$ does not converge to a_0. In §2 some theorems are proved assuming uniform convergence of $\{\rho_n\}_{n=1}^{\infty}$ to ρ_0. The example in §1 shows that none of the results in §2 remains valid if uniform convergence of the metrics is replaced by pointwise convergence. In §3 a fixed point theorem for compact nonempty set-valued contractive mappings is proved and it is shown that the analogous statement for closed and bounded nonempty set-valued contractive mappings is false. It is then indicated how the results of §2 can be extended to compact nonempty set-valued contractive mappings.

Let (X, ρ) be a metric space. A function $f: X \rightarrow X$ is said to be a *ρ-contraction* if and only if there exists λ, $0 \leqq \lambda < 1$, such that $\rho(f(x), f(y)) \leqq \lambda\rho(x, y)$ for all $x, y \in X$(λ is called a *ρ-Lipschitz constant* for f). A function $f: X \rightarrow X$ is said to be *ρ-contractive* if and only if $\rho(f(x), f(y)) < \rho(x, y)$ for all $x, y \in X$, $x \neq y$.

The following theorem was proved in [4].

THEOREM A. *Let (X, ρ) be a locally compact metric space, let $f_n: X \rightarrow X$ be a ρ-contraction with fixed point a_n for each $n = 1, 2, \cdots$, and let $f_0: X \rightarrow X$ be a ρ-contraction with fixed point a_0. If the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f_0, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.*

In [5] it was shown that closed and bounded nonempty set-valued contraction mappings defined on a complete space have fixed points. Theorem A and other results in [4] were generalized to compact nonempty set-valued contractions.

Throughout this paper two metrics, d_1 and d_2, for the same set

X will be called *equivalent* if and only if the identity mapping from (X, d_1) to (X, d_2) is a homeomorphism.

1. The example. Let $X = \{(2^{-i}, 2^{-j}) \mid i, j = 0, 1, 2, \cdots, \infty\}$ with the convention that $2^{-\infty} = 0$. Let $x, y \in X$ and assume $x = (2^{-k}, 2^{-l})$ and $y = (2^{-m}, 2^{-p})$. For each integer $n > 0$ let

$$\rho_n(x, y) = \rho_n(y, x) = \begin{cases} |2^{-k} - 2^{-m}| & , \text{ if } l = p = n, \\ 2 - 2^{-k} + 2^{-p}, & \text{if } l = n, p \neq n \text{ and } m = 0, \\ 4, & \text{if } l = n, p \neq n \text{ and } m \neq 0, \\ |2^{-l} - 2^{-p}|, & \text{if } l \neq n, p \neq n \text{ and } m = k = 0, \\ 4 & \text{if } l \neq n, p \neq n, m = 0 \text{ and } k \neq O, \\ |2^{-k} - 2^{-m}| + |2^{-l} - 2^{-p}|, & \text{if } l \neq n, p \neq n, m \neq 0 \text{ and } k \neq O; \end{cases}$$

and let

$$\rho_0(x, y) = \rho_0(y, x) = \begin{cases} 4, & \text{if } k = 0 \text{ and } m \neq 0, \\ |2^{-k} - 2^{-m}| + |2^{-l} - 2^{-p}|, & \text{if } k \neq 0 \text{ and } m \neq 0, \\ |2^{-l} - 2^{-p}| & , \text{ if } k = 0 \text{ and } m = 0 . \end{cases}$$

It is easy to verify that, for each integer $n \geqq 0$, ρ_n is a metric on X which is equivalent to the metric on X inherited from the plane. Furthermore, $\{\rho_n\}_{n=1}^{\infty}$ converges pointwise to ρ_0.

For each integer $n > 0$ let $f_n: X \to X$ be given by

$$f_n(2^{-i}, 2^{-j}) = \begin{cases} (2^{-(i+1)}, 2^{-n}), & \text{if } j = n , \\ (1, 2^{-n}) & , \text{ if } j \neq n \text{ and } i = 0 , \\ (1, 0) & , \text{ if } j \neq n \text{ and } i \neq 0 . \end{cases}$$

Define $f_0: X \to X$ by $f_0(x) = (1, 0)$ for all $x \in X$. It is easy to prove that

(1) for each $n \geqq 0, f_n$ is a ρ_n-contraction mapping with ρ_n-Lipschitz constant equal to $1/2$;

(2) the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f_0;

(3) the mapping f_n has fixed point

$$a_n = \begin{cases} (0, 2^{-n}), & n > 0 , \\ (1, 0), & n = 0 ; \end{cases}$$

(4) the sequence $\{a_n\}_{n=1}^{\infty}$ of fixed points converges to $(0, 0)$ and not to the fixed point $(1, 0)$ of the limit function f_0.

2. Theorems for single-valued mappings. The following lemma will be useful.

LEMMA 1. *Let (X, ρ_0) be a metric space and K a compact subset of X. Let $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of metrics converging uniformly to ρ_0 such that each ρ_n is equivalent to ρ_0, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of ρ_n-contractive mappings converging pointwise on X to a function f_0. Then the sequence $\{f_n\}_{n=1}^{\infty}$ converges (ρ_0-) uniformly on K to f_0.*

Proof. Let $\eta > 0$ and choose $\delta = \eta/3$. Let N be a natural number such that if $n > N$ then $|\rho_n(x, y) - \rho_0(x, y)| < \delta$ for all $x, y \in X$. If $n > N$ and $x, y \in X$ such that $\rho_0(x, y) < \delta$, then

$$\begin{aligned}\rho_0(f_n(x), f_n(y)) &< \delta + \rho_n(f_n(x), f_n(y)) < \delta + \rho_n(x, y) \\ &< \delta + \delta + \rho_0(x, y) < 3\delta = \eta .\end{aligned}$$

Hence if $x, y \in X$ and $\rho_0(x, y) < \delta$, then $\rho_0(f_n(x), f_n(y)) < \eta$ for all $n > N$. Since $f_1, f_2, \cdots, f_N$ are each (ρ_0-) uniformly continuous on K, the sequence $\{f_n\}_{n=1}^{\infty}$ is (ρ_0-) equicontinuous on K. Therefore, since K is compact and $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f_0, it follows that $\{f_n\}_{n=1}^{\infty}$ converges (ρ_0-) uniformly on K to f_0. This completes the proof of the lemma.

REMARK. Under the conditions of Lemma 1, without assuming ρ_n is equivalent to ρ_0, it is not difficult to prove that f_0 is ρ_0-nonexpansive. Let $x, y \in X$ and let $\varepsilon > 0$. Choose N such that if $n \geqq N$ then ρ_n is uniformly within $\varepsilon/4$ of ρ_0, $\rho_0(f_n(x), f_0(x)) < \varepsilon/4$, and $\rho_0(f_n(y), f_0(y)) < \varepsilon/4$. Then, if $n \geqq N$,

$$\begin{aligned}&\rho_0(f_0(x), f_0(y)) \leqq \rho_0(f_0(x), f_n(x)) + \rho_0(f_n(x), f_n(y)) \\ &+ \rho_0(f_n(y), f_0(y)) < \varepsilon/4 + \rho_n(f_n(x), f_n(y)) + \varepsilon/4 + \varepsilon/4 \\ &< \rho_n(x, y) + 3\varepsilon/4 < \rho_0(x, y) + \varepsilon .\end{aligned}$$

Since ε was arbitrary, f_0 is ρ_0-nonexpansive.

The following theorem is a generalization of Theorem A, even in the case when each $\rho_n = \rho_0$, because each f_n for $n = 0, 1, 2, \cdots$ is assumed to be only contractive.

THEOREM 1. *Let (X, ρ_0) be a locally compact metric space and assume $\{\rho_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$ satisfy the hypotheses of Lemma 1. If f_0 is ρ_0-contractive and f_n has fixed point a_n for each $n = 0, 1, 2, \cdots$, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.*

Proof. Let $\varepsilon > 0$ be chosen such that $K(a_0, \varepsilon) = \{x \in X \mid \rho_0(a_0, x) \leqq \varepsilon\}$ is a compact subset of X. Since Lemma 1 applies, $\{f_n\}_{n=1}^\infty$ converges uniformly on $K(a_0, \varepsilon)$ to f_0. Choose a natural number N such that if $n \geqq N$ and $x \in K(a_0, \varepsilon)$, then

$$\rho_0(f_n(x), f_0(x)) < \mu = \varepsilon - \sup\{\rho_0(f_0(z), a_0) \mid z \in K(a_0, \varepsilon)\}$$

(clearly $\mu > 0$ by the ρ_0-contractiveness of f_0 and the compactness of $K(a_0, \varepsilon)$). Then, if $n \geqq N$ and $x \in K(a_0, \varepsilon)$,

$$\begin{aligned}\rho_0(f_n(x), a_0) &\leqq \rho_0(f_n(x), f_0(x)) + \rho_0(f_0(x), a_0)\\ &< \mu + \rho_0(f_0(x), a_0) \leqq \mu + (\varepsilon - \mu) = \varepsilon .\end{aligned}$$

Thus, for $n \geqq N$, f_n maps $K(a_0, \varepsilon)$ into itself. Letting g_n be the restriction of f_n to $K(a_0, \varepsilon)$ for each $n \geqq N$, we see that g_n is a ρ_n-contractive mapping of the (ρ_n-) compact metric space $K(a_0, \varepsilon)$ into itself. Therefore, g_n has a fixed point in $K(a_0, \varepsilon)$ for each $n \geqq N$ [1] which must, from the definition of g_n and the fact that f_n has only one fixed point, be a_n. Hence, $a_n \in K(a_0, \varepsilon)$ for each $n \geqq N$. It follows that the sequence $\{a_n\}_{n=1}^\infty$ converges to a_0.

THEOREM 2. *Let (X, ρ_0) be a metric space and assume $\{\rho_n\}_{n=1}^\infty$ and $\{f_n\}_{n=0}^\infty$ satisfy the hypotheses of Lemma 1. If f_n has fixed point a_n for each $n = 1, 2, \cdots$ and some subsequence of $\{a_n\}_{n=1}^\infty$ converges to a point $x_0 \in X$, then x_0 is a fixed point of f_0. In particular, if f_0 is ρ_0-contractive with (unique) fixed point a_0, then $x_0 = a_0$.*

Proof. Let $\{a_{n_i}\}_{i=1}^\infty$ be a subsequence of $\{a_n\}_{n=1}^\infty$ such that $\{a_{n_i}\}_{i=1}^\infty$ converges to a point $x_0 \in X$. Applying Lemma 1 to $K = \{x_0, a_{n_1}, a_{n_2}, \cdots\}$, we see that $\{f_{n_i}\}_{i=1}^\infty$ converges uniformly on K to f_0. Hence, $\{f_{n_i}(a_{n_i})\}_{i=1}^\infty$ converges to $f_0(x_0)$. But, since $f_{n_i}(a_{n_i}) = a_{n_i}$ for each $i = 1, 2, \cdots$ and since $\{a_{n_i}\}_{i=1}^\infty$ converges to x_0, this proves that $f_0(x_0) = x_0$.

THEOREM 3. *Let (X, ρ_0) be a metric space, let $\{\rho_n\}_{n=1}^\infty$ be a sequence of metrics converging uniformly to ρ_0, and let $\{f_n\}_{n=1}^\infty$, each f_n having fixed point a_n, be a sequence of ρ_n-contractions converging ρ_0-pointwise on X to a function f_0 with fixed point a_0. If there exists $\beta < 1$ such that, for each $n = 1, 2, \cdots, \beta$ is a ρ_n-Lipschitz constant for f_n, then the sequence $\{a_n\}_{n=1}^\infty$ converges to a_0.*

Proof. Let $\varepsilon > 0$. Choose a natural number N such that if $n \geqq N$, then

$$|\rho_n(x, y) - \rho_0(x, y)| < \frac{1 - \beta}{2 + \beta} \cdot \varepsilon$$

for all $x, y \in X$ and

$$\rho_0(f_n(a_0), a_0) < \frac{1-\beta}{2+\beta}\cdot\varepsilon .$$

Then, for $n \geqq N$,

$$\begin{aligned}\rho_0(a_n, a_0) &\leqq \rho_0(f_n(a_n), f_n(a_0)) + \rho_0(f_n(a_0), f_0(a_0)) < \rho_n(f_n(a_n), f_n(a_0)) \\ &+ \frac{2(1-\beta)}{2+\beta}\cdot\varepsilon \leqq \beta\rho_n(a_n, a_0) + \frac{2(1-\beta)}{2+\beta}\cdot\varepsilon < \beta\rho_0(a_n, a_0) \\ &+ \frac{\beta(1-\beta)}{2+\beta}\cdot\varepsilon + \frac{2(1-\beta)}{2+\beta}\cdot\varepsilon = \beta\rho_0(a_n, a_0) + (1-\beta)\cdot\varepsilon .\end{aligned}$$

Hence, if $n \geqq N$, $\rho_0(a_n, a_0) < \epsilon$, proving that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.

REMARK. Using the techniques in the remark following Lemma 1, it can be shown that the function f_0 in Theorem 3 is a ρ_0-contraction with ρ_0-Lipschitz constant β.

REMARK. In the proof of Theorem 1 we showed that all but finitely many of the functions f_n mapped the compact set $K(a_0, \varepsilon)$ into itself. We could have concluded (without the assumption that each f_n had a fixed point) from Edelstein's Theorem 1 [1] that all the functions f_n mapping $K(a_0, \varepsilon)$ into itself had fixed points. Furthermore, this procedure would prove that these fixed points converge to a_0.

3. Theorems for multi-valued mappings.

Let (X, ρ) be a metric space. By $2^X[CB(X)]$ we mean the space of all compact [closed and bounded] nonempty subsets of X metrized by H, the Hausdorff metric induced by ρ[2, p. 131]. A function $F: X \to CB(X)$ is said to be a *multi-valued ρ-contraction* [*ρ-contractive*] *mapping* if and only if there exists $\lambda < 1$ such that $H(F(x), F(y)) \leqq \lambda\rho(x, y)$ for all x, $y \in X[H(F(x), F(y)) < \rho(x, y)$ for all $x, y \in X$ with $x \neq y]$. A point $x \in X$ is said to be a *fixed point* for a function $F: X \to CB(X)$ if and only if $x \in F(x)$ (see [5] for further discussion).

Let F be a continuous function from X into 2^X. If A is in 2^X, then $\cup\{F(a) \mid a \in A\}$ is also in 2^X[3, p. 168]. The function $\hat{F}: 2^X \to 2^X$ defined by $\hat{F}(A) = \cup\{F(a) \mid a \in A\}$ for each $A \in 2^X$ is called the *function induced by F*. (A function similarly defined on $CB(X)$ would not necessarily have values in $CB(X)$. Occasionally, when this more generally defined induced function has all its values in $CB(X)$, we will use it and denote it by $\hat{F}$.) It is easy to see that the continuity of F implies the continuity of $\hat{F}$. If F is a multi-valued ρ-contraction or ρ-contractive mapping, then $\hat{F}$ is a ρ-contraction [5] or (respectively, on 2^X) a ρ-contractive mapping.

The next theorem is an extension of Theorem 1 in [1] and is closely related to Theorem 5 in [5]. Since the existence of fixed points is hypothesized in each of the theorems in §2, it is not necessary to include the next theorem in order to generalize the results of §2 to set-valued mappings. However, it is included to show that restrictions similar to those imposed on single-valued contractive mappings guarantee that compact nonempty set-valued contractive mappings have fixed points.

THEOREM 4. *Let (X, ρ) be a metric space and let $F: X \to 2^X$ be a multi-valued ρ-contractive mapping. If there exists $A \in 2^X$ such that some subsequence of the sequence $\{\hat{F}^n(A)\}_{n=1}^\infty$ of iterates of $\hat{F}$ at A converges to a member of 2^X, then F has a fixed point.*

Proof. Let $F: X \to 2^X$ be a multi-valued ρ-contractive mapping and let $A \in 2^X$ such that a subsequence $\{\hat{F}^{n_i}(A)\}_{i=1}^\infty$ of $\{\hat{F}^n(A)\}_{n=1}^\infty$ converges to a set $B \in 2^X$. Now, since $\hat{F}$ is a ρ-contractive mapping (see comments above), we may apply Theorem 1 of [1] and obtain that B is a fixed point of $\hat{F}$, i.e., $\hat{F}(B) = B$. Define a real-valued continuous function g on B by $g(x) = \inf\{\rho(x, y) \mid y \in F(x)\}$ for each $x \in B$. Since B is compact, g assumes its minimum r at some point $b \in B$. Suppose $r > 0$. Since $F(b)$ is compact, there is a point $z \in F(b)$ such that $g(b) = \rho(b, z)$. Because $g(b) = r > 0, b \neq z$; also, since $z \in F(b)$, $g(z) \leqq H(F(b), F(z))$. It follows that

$$g(z) \leqq H(F(b), F(z)) < \rho(b, z) = g(b), \text{ i.e., } g(z) < g(b).$$

However, since $\hat{F}(B) = B$, $z \in B$ and this contradicts the minimality of g at b. Hence, $r = 0$. It now follows that $b \in F(b)$, which proves the theorem.

REMARK. Let F and A satisfy the hypotheses of Theorem 4 and let $\{\hat{F}^{n_i}(A)\}_{i=1}^\infty$ be a convergent subsequence of $\{\hat{F}^n(A)\}_{n=1}^\infty$ such that $\lim_{i\to\infty} \hat{F}^{n_i}(A) = B \in 2^X$. Then

(1) $\{\hat{F}^n(A)\}_{n=1}^\infty$ converges to B and

(2) there exists a point $p \in X$ such that $\{\hat{F}^n(\{p\})\}_{n=1}^\infty$ converges to B.

The proof of (1) is the same as the argument in Remark 3.2 of [1]. To prove (2), choose $p \in B$. Then, since 2^B is compact and $\hat{F}^n(\{p\}) \in 2^B$ for each $n = 1, 2, \cdots$, the sequence $\{\hat{F}^n(\{p\})\}_{n=1}^\infty$ has a convergent subsequence which must, since it converges to the unique fixed point of $\hat{F}[1]$, converge to B. By (1), with $\{p\}$ playing the role of A, it now follows that $\{\hat{F}^n(\{p\})\}_{n=1}^\infty$ converges to B.

REMARK. Let $\hat{F}$ and B be as in the proof of Theorem 4, i.e., $\hat{F}(B) = B$. We proved that there is a fixed point of F in B. Later we shall use the fact that *every* fixed point of F is in B and we now verify this. Let x_0 be a fixed point of F and suppose $x_0 \notin B$. Then $\inf\{\rho(x_0, y) \mid y \in B\} = \eta > 0$ and, since B is compact, there is $y_0 \in B$ such that $\rho(x_0, y_0) = \eta$. Since $H(F(x_0), F(y_0)) < \rho(x_0, y_0) = \eta$ and $x_0 \in F(x_0)$, there is $z_0 \in F(y_0)$ such that $\rho(x_0, z_0) < \eta$. But, since $\hat{F}(B) = B$ and $y_0 \in B$, $F(y_0) \subset B$ which implies that $z_0 \in B$. This contradicts the definition of η.

Since a multi-valued contradiction mapping on a complete space into $CB(X)$ has a fixed point [5], one might conjecture that Theorem 4 could be extended to multi-valued contractive mappings into $CB(X)$ (assuming $\hat{F}$ maps into $CB(X)$). This is not possible in general, as is seen in the following

EXAMPLE. Let $X = \{x_n \mid n = 0, \pm 1, \pm 2, \cdots\} \cup \{y\}$ be a countable set of distinct points and define a metric ρ for X by the conditions

(1) $\rho(x_n, x_n) = \rho(y, y) = 0$ for all $n = 0, \pm 1, \pm 2, \cdots$;

(2) $\rho(x_n, y) = \rho(y, x_n) = 10$ for all $n = 0, \pm 1, \pm 2, \cdots$;

and (3) $\rho(x_n, x_m) = \gamma_n + \gamma_m$ for $x_n, x_m \in X$ with $x_n \neq x_m$, where

$$\gamma_k = \frac{1}{4} + \frac{1}{2^k} \quad \text{if} \quad k > 0 \quad \text{and} \quad \gamma_k = 2 + \frac{1}{k-1} \quad \text{if} \quad k \leqq 0 .$$

It is easy to verify that ρ is a metric. Define $F: X \to CB(X)$ by letting $F(y) = X - \{y\}$ and $F(x_n) = x_{n+1}$ for each $x_n \in X$. It is easy to see that F is a multi-valued ρ-contractive mapping and that $\hat{F}$ maps $CB(X)$ into $CB(X)$. Since $\hat{F}^n(\{y\}) = X - \{y\}$ for each $n = 1, 2, \cdots$, it is obvious that the sequence $\{\hat{F}^n(\{y\})\}_{n=1}^\infty$ converges (to $X - \{y\}$). However, F has no fixed point. It is interesting to note that, though $\hat{F}$ maps $CB(X)$ into $CB(X)$, $\hat{F}$ is not a contractive mapping. Also note that (X, ρ) is complete.

We now present the types of modifications necessary to obtain generalizations of the results in §2 to multi-valued mappings. For the remainder of this section H_n will denote the Hausdorff metric for 2^X induced by ρ_n for $n = 0, 1, 2, \cdots$.

It is well-known that equivalent metrics for X may not induce equivalent Hausdorff metrics for $CB(X)$ [2, p. 131]. However, equivalent metrics for X do induce equivalent metrics for 2^X. We need the following lemma.

LEMMA 2. *If ρ_1 and ρ_2 are equivalent metrics for X, then H_1 and H_2 are equivalent metrics for 2^X.*

Proof. Let $\{A_i\}_{i=1}^\infty$ be a sequence in 2^X such that $\{A_i\}_{i=1}^\infty$ converges, with respect to H_1, to a set $A \in 2^X$. It follows that $B = (\bigcup_{i=1}^\infty A_i) \cup A$ is a compact subset of X[3, p. 168]. Hence, $\rho_1|_B$ is uniformly equivalent to $\rho_2|_B$. Therefore, since $H_n|_{2^B}$ is the Hausdorff metric induced by $\rho_n|_B$ for each $n = 1$ or 2, it now follows that $H_1|_{2^B}$ is equivalent to $H_2|_{2^B}$. Thus, the sequence $\{A_i\}_{i=1}^\infty$ converges to A with respect to H_2. By symmetry we obtain the desired result (cf. 4).

Now we show how to generalize Theorem 1 of §2.

Let (X, ρ_0) be a locally compact metric space and let $\{\rho_n\}_{n=1}^\infty$ be a sequence of metrics for X converging uniformly on X to ρ_0 such that each ρ_n is equivalent to ρ_0. For each $n = 0, 1, 2, \cdots$, let $F_n: X \to 2^X$ be a multi-valued ρ_n-contractive mapping with the property that there is a set $A_n \in 2^X$ such that some subsequence of $\{\hat{F}_n^k(A_n)\}_{k=1}^\infty$ converges to a member of 2^X or, equivalently, that $\hat{F}_n$ has a fixed point $B_n \in 2^X$ (that such an hypothesis is necessary is discussed in the remark below). Let a_n be a fixed point of F_n for each $n = 1, 2, \cdots$ (actually, a_n exists for each n by Theorem 4 above).

Suppose the sequence $\{F_n\}_{n=1}^\infty$ converges pointwise on X to F_0. By Lemma 2, H_n is equivalent to H_0 for each $n = 1, 2, \cdots$. Routine computations show that the sequence $\{H_n\}_{n=1}^\infty$ converges uniformly on 2^X to H_0. A slight modification of the proof of Lemma 1 shows that $\{F_n\}_{n=1}^\infty$ converges (H_0-) uniformly on compact subsets of X to F_0. This implies that $\{\hat{F}_n\}_{n=1}^\infty$ converges pointwise on 2^X to $\hat{F}_0$. Since $(2^X, H_0)$ is locally compact, we can now apply Theorem 1 to the sequence $\{B_n\}_{n=1}^\infty$ and conclude that $\{B_n\}_{n=1}^\infty$ converges to B_0. Since $a_n \in B_n$ for each $n = 1, 2, \cdots$ (see the 2nd remark following Theorem 4), $\{a_n\}_{n=1}^\infty$ is a sequence of points in the compact set $\cup \{B_n \mid n = 0, 1, 2, \cdots\}$. Hence, $\{a_n\}_{n=1}^\infty$ has a convergent subsequence which, by an easy modification of Lemma 3 of [5], must converge to a fixed point of F_0(note that not every point in B_0 is necessarily a fixed point of F_0).

REMARK. The restriction above that $\hat{F}_n$ have a fixed point in 2^X was necessary (even in the case where X is complete; compare with Theorem 9 [5]). To see this let

$$X = \left\{x_i \mid x_i = i + 1 + \frac{i}{i+1} \quad \text{for each} \quad i = 0, 1, 2, \cdots\right\}$$

with absolute value distance. Define $F: X \to 2^X$ by

$$F(x_i) = \{x_0, x_1, \cdots, x_i + 1\} \quad \text{for each} \quad i = 0, 1, 2, \cdots$$

It is easy to verify that F is a multi-valued contractive mapping and that $\hat{F}: 2^X \to 2^X$ has no fixed point. Each point of X is a fixed point of F. If we let

$$F_n = F \quad \text{and} \quad a_n = n + 1 + \frac{n}{n+1}$$

for each $n = 0, 1, 2, \cdots$, then we see that $\{F_n\}_{n=1}^{\infty}$ converges to F_0 but $\{a_n\}_{n=1}^{\infty}$ has no convergent subsequence.

The modifications of other theorems in §2 are carried out in an analogous fashion.

4. *Added in proof.* This result is contained in [3] as Theorem 3.3.

BIBLIOGRAPHY

1. M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74-79.
2. J. L. Kelley, *General topology*, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1959.
3. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152-182.
4. S. B. Nadler, Jr., *Sequences of contractions and fixed points*, Pacific J. Math. **27** (1968), 579-585.
5. ———, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475-488.

Received January 22, 1969, and in revised form April 23, 1969.

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LOUISIANA