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ALGEBRAS FOR WHICH EVERY INDECOMPOSABLE RIGHT MODULE IS INVARIANT IN ITS INJECTIVE ENVELOPE

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The algebras of the title are characterized as those having Tachikawa's strong left cyclic representation type (SLCRT) with the further property that every quotient of a principal indecomposable left module has square free socle. Moreover it turns out that these are precisely the algebras for which every indecomposable right module is quasi-injective in the sense of Johnson and Wong, and that these algebras have finite module type.

Throughout, the letter A denotes a finite dimensional algebra with radical N over a field K with more than two elements. All modules considered are unitary finitely generated A-modules. A completely reducible module M is said to be *square free* if the homogeneous components of M are simple. If M is any module the *socle* of M is the annihilator of N in M and the *top* of M is the factor module $M/\text{Rad } M$.

Johnson and Wong [3] have introduced the concept of a quasi-injective module, which is characterized as one that is carried into itself by all endomorphisms of its injective envelope. Tachikawa [5] has introduced algebras of *strong left cyclic representation type* (SLCRT). An algebra has SLCRT in case each of its indecomposable left modules is a quotient of a principal indecomposable (equivalently, has a simple top). He has given four ideal-theoretic conditions which, together, are necessary and sufficient for an algebra to have SLCRT. Our main result places the algebras of the title among those having SLCRT and distinguishes them by an additional condition on the left principal indecomposables. This condition is precisely the Curtis-Jans [1] condition that socles of indecomposable modules be square free, which yields algebras of *finite module type* (=those having a finite number of isomorphism classes of indecomposable modules) when the base field is algebraically closed. An algebra satisfying the conditions of the theorem is said to be of *right invariant module type*. We give an example to show that not all SLCRT algebras are of right invariant module type.

If M is a quasi-injective module over any ring R (equivalently, M is stable under all R-endomorphisms of $E(M)$ [3, Th. 1.1]) then, as observed by Wu and Jans on page 442 of [6], M decomposes whenever $E(M)$ does. Thus if M is an indecomposable quasi-injective R-

module with an essential socle then M is an R-End $(E(S))$ submodule of $E(S)$ where S is the (necessarily) simple socle of M. In the case where M is an A-module this observation and [6, Th. 3.1] are equivalent via the vector space dual. In the following lemma we show that for indecomposable A-modules a stronger result (and, of course, its dual) holds - namely, in order for an indecomposable A-module to be quasi-injective it is sufficient that it merely be invariant under the *automorphisms* of its injective envelope. The key to the proof is to show that an A-module satisfying this condition must have a simple socle.

LEMMA. *If an indecomposable A-module M is invariant under the automorphisms of its injective envelope $E(M)$ then M is quasi-injective.*

Proof. Suppose M does not have a simple socle. Let $M_1 \oplus M_2$ be maximal with respect to being a decomposable submodule of M. Then each M_i $(i = 1,2)$ is maximal with respect to $M_1 \cap M_2 = 0$. It follows that $(M_1 \oplus M_2)/M_i$ is essential in M/M_i $(i = 1, 2)$ and so the module $(M_1 \oplus M_2)/M_1 \oplus (M_1 \oplus M_2)/M_2$ is essential in $M/M_1 \oplus M/M_2$. But the diagonal copy of M, namely $M_d = \{(m + M_1, m + M_2) \mid m \in M\}$, is situated between the above two modules and hence is also essential in $M/M_1 \oplus M/M_2$. Let $k \neq 1 \in K$ be a non-zero scalar, then the automorphism of the direct sum that takes the pair $(m + M_1, m + M_2)$ to $(km + M_1, m + M_1)$ maps the copy M_d into a different submodule of the direct sum, which is easily checked. It follows that M is not invariant under the automorphisms of its injective envelope unless M has simple socle. Thus if M is invariant under the automorphisms of its injective envelope we see that $E(M)$ is indecomposable and has a completely primary endomorphism ring R. If $\alpha \in R$ then either α or $1 - \alpha$ is an automorphism. In either case $\alpha(M) \subseteqq M$ so M is quasi-injective.

THEOREM. *The following statements about the algebra A are equivalent:*

(a) *Each indecomposable right A-module is carried into itself under the action of any automorphism of its injective envelope.*

(b) *Each indecomposable right A-module is carried into itself under the action of any endomorphism of its injective envelope* (*i.e.*, *is quasi-injective*).

(c) *Each indecomposable left* (*right*) *A-module has simple top* (*socle*) *and square free socle* (*top*).

Proof. The parenthetical version of (c) is equivalent to its nonparenthetical version by properties of the K-dual. We shall use the former in the proof.

(a) $\Leftarrow$ (b) This is a consequence of the lemma.

(b) $\Rightarrow$ (c) Since an indecomposable quasi-injective has simple socle, we see that each right principal indecomposable is uniserial whenever (b) is satisfied. Suppose this is the case but (c) does not hold. Then there is an indecomposable injective right module E with submodules xeA and yeA ($x, y \in E$, e a primitive idempotent in A) such that $xeA \not\subseteq yeA$ and $yeA \not\subseteq xeA$. But since eA is uniserial, one of these is an epimorph of the other. This epimorphism, extended to an endomorphism of E, contradicts our assumption.

(c) $\Rightarrow$ (b) Assume (c) and suppose some indecomposable module M is not mapped into itself by an endomorphism φ of its injective envelope $E(S)$, where S is the simple socle of M. Then there is a submodule $M' \subseteq E(S)$ that is minimal with respect to $\varphi(M') \not\subseteq M'$. Now $\varphi(M'N) \subseteq M'N$, so that the top of the indecomposable module $M' + \varphi(M')$ is $(M' + \varphi(M'))/(M' + \varphi(M'))N = (M' + \varphi(M'))/M'N$ which is not square free. This contradiction completes the proof.

If K is an algebraically closed field Curtis and Jans [1] have shown that A is of finite module type if each indecomposable left A-module has a square free socle. However, even if K is not algebraically closed, we have the

COROLLARY. *If the algebra A is of right invariant module type then A has finite module type.*

Proof. If the submodule lattice of a principal indecomposable Ae is infinite then it is nondistributive and modular, and hence contains a projective root (see [2, p. 419]), which introduces a quotient of Ae with non-square free socle.

We do not know whether an arbitrary SLCRT algebra has finite module type.[1]

REMARKS. (i) If K is infinite, statement (c) of the theorem can be replaced by the demand that each indecomposable left module be a quotient of a principal indecomposable which has a finite submodule lattice.

(ii) One of Tachikawa's conditions states that an SLCRT algebra is right generalized uniserial. Thus any quasi-Frobenius algebra of right invariant module type is generalized uniserial.

[1] As this goes to press we note that A. V. Roiter's paper has appeared, Izv. Akad. Nauk S.S.S.R. Ser. Mat. **32** (1968), 1275-1282, in which the Brauer-Thrall conjecture is proved, answering this in the affirmative.

(iii) Singh and Jain [4] have recently defined a pseudo-injective module as a module M for which every monomorphism from a submodule of M into M can be extended to an endomorphism of M. They have shown that in certain special cases pseudo-injective modules are also quasi-injective. We note that by virtually copying the necessity part of the proof of [3, Th. 1.1] and the sufficiency part of [4, Th. 3.7] one can show that a module with a finitely generated essential socle is pseudo-injective if and only if it is invariant under the automorphisms of its injective envelope.

EXAMPLES. (i) Let A be the algeba of matrices of the form

$$\begin{bmatrix} x & 0 & 0 \\ u & x & 0 \\ v & w & y \end{bmatrix}$$

with entries in any field K. This is an SLCRT algebra, as can be verified by checking Tachikawa's four conditions [5]. However the socle of the principal indecomposable Ae ($e = e_{11} + e_{22}$ with e_{ij} the ij-th matrix unit) is not square free, so A does not have right invariant module type.

(ii) The subalgebra A' of A obtained by setting $w = 0$ still satisfies Tachikawa's conditions and it is easily checked that every quotient of $A'e$ has square free socle. The only other principal indecomposable is $A'e_{33}$ which is already simple. Thus A' is an example of an algebra of right invariant module type that is not generalized uniserial.

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