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SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN DENSITY II

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Let S denote the set of all infinite increasing sequences of positive integers. For all $A \cong \{a_n\}$ and $B = \{b_n\}$ in S define the metric $\rho(A, B) = 0$ if $A = B$; i.e., if $a_n = b_n$ for all n and $\rho(A, B) = 1/k$ otherwise, where k is the smallest value of n for which $a_n \neq b_n$. The main object of this note is to show that the set of points of continuity of the Schnirelmann density $d(A)$ is a residual set and that this is the best possible result of this type.

The space S and some of the properties of densities defined on it have been discussed previously [2, 3, 4]. In particular, it has been shown that the set of points of continuity of $d(A)$ is the set of all points having density zero. Let $L_a = \{A \in S \mid d(A) = a\} (0 \leqq a \leqq 1)$ denote the level sets of $d(A)$ and define $M_a = \{A \in S \mid d(A) \geqq a\}$. Then $\bar{L}_a = M_a$ so that M_a is closed and L_a is dense in M_a [4]. These results are required in the sequel. A brief and lucid account of all other necessary topological results is given in [1].

THEOREM 1. *The family of all sets of the form* $S(m, n) = \{A \in S \mid a_n = m\}$ *is a sub-basis for the topology of* S.

Proof. If $A \in S(m, n)$ and $B \notin S(m, n)$, then $\rho(A, B) \geqq 1/n$. Hence $S - S(m, n)$ is closed and $S(m, n)$ is open. Also, the spheres $S_\varepsilon(A) = \{B \in S \mid \rho(A, B) < \varepsilon\}$, $0 < \varepsilon \leqq 1$, constitute a basis for S and the desired result follows since

$$S_\varepsilon(A) = \bigcap_{n=1}^{[1/\varepsilon]} S(a_n, n) .$$

COROLLARY. *S has a countable basis.*

COROLLARY. *S is separable.*

It is also clear that S is a subspace of $\times_{n=1}^{\infty} P_n$, where P_n is the set of all positive integers with the discrete topology for each n.

THEOREM 2. *S is complete.*

Proof. Let $A_n = \{a_{n,\nu}\}_{\nu=1}^{\infty}$ and suppose that $\{A_n\}$ is a Cauchy sequence in S. Also, let n_k be the smallest positive integer such that

$\rho(A_m, A_n) < 1/k$ for all $m, n \geqq n_k$ and define $A = \{a_{n_k,k}\}_{k=1}^{\infty}$. Since all of the A_n's have the same first k terms for $n \geqq n_k$, it is clear that $A \in S$ and $\rho(A_n, A) < 1/k$ for all $n \geqq n_k$. Hence $\lim_{n\to\infty} \rho(A_n, A) = 0$ and S is complete.

The following corollaries are a consequence of the Baire category theorem and the fact that M_a is a closed subset of S.

COROLLARY. *M_a is complete.*

COROLLARY. *M_a is a set of the second category in itself.*

The following result would be of no interest for those values of a for which the second of the above corollararies fails to hold.

THEOREM 3. *L_a is residual in M_a.*

Proof. $M_a - L_a = \bigcup_{k=1}^{\infty} M_{a+1/k}$. Since $\bar{L}_a = M_a$, L_a is dense in M_a and, since $M_{a+1/k} \subset M_a$, L_a is dense in $M_{a+1/k}$. Also, since $M_{a+1/k}$ is closed, $M_{a+1/k}$ is nowhere dense in M_a and $M_a - L_a$ is a set of the first category in M_a.

Since the set of points of continuity of $d(A)$ is L_0 and $M_0 = S$, the following result ensues.

COROLLARY. *The set of points of continuity of $d(A)$ is residual in S.*

The following theorem shows that the above corollary is a best possible result in the following sense. In the true statement, $S - L_0$ is a countable union of nowhere dense sets, the word countable can not be replaced by finite.

THEOREM 4. *$\overline{M_a - L_a}$ is open if and only if $a = 0$ or 1.*

Proof. $\overline{M_1 - L_1}$ is the empty set and hence open. Also, it is easily seen that $\overline{M_0 - L_0} = S(1, 1)$ in the notation of Theorem 1 and hence open.

Suppose that $\overline{M_a - L_a}$ is open for $a > 0$. Then $\overline{M_a - L_a} \subset M_a$, since M_a is closed, and it follows that $L_0 \subset S - \overline{M_a - L_a}$. Since $\bar{L}_0 = S$ and $S - \overline{M_a - L_a}$ is closed, we have $S - \overline{M_a - L_a} = S$ and $\overline{M_a - L_a}$ is the empty set. Thus $a = 1$ and the proof is complete.

The following result is included in the preceding proof.

COROLLARY. *The support of* $d\ (A)$ *is the set of all sequences with first term one.*

The final result concerns the asymptotic density

$$\delta(A) = \liminf A(k)/k\ ,$$

where $A(k)$ denotes the number of elements of A which do not exceed k.

THEOREM 5. $\delta(A)$ *is a function of Baire class two.*

Proof. Let $\delta_n(A) = \inf_{k \geqq n} A(k)/k$. Then $\delta_n(A)$ is a function of Baire class one [4, Th. 3]. Also, $\delta(A) = \lim_{n \to \infty} \delta_n(A)$. Now $\delta(A)$ is obviously everywhere discontinuous on S. Suppose $\delta(A)$ is a function of Baire class one. Then the set of points of discontinuity of $\delta(A)$ is a set of the first category [5, Th. 36]. But S is a set of the second category and the desired result follows.

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