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TORSION CLASSES AND PURE SUBGROUPS

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In this note we obtain a classification of the classes $\mathscr{T}$ of abelian groups satisfying the following closure conditions:
(i) If $\{A_\mu \mid \mu \in M\} \subseteq \mathscr{T}$, then $\mathscr{T}$ contains the direct sum $\sum A_\mu$.
For a short exact sequence

$$(*) \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

(ii) $C \in \mathscr{T}$ if $B \in \mathscr{T}$
(iii) $B \in \mathscr{T}$ if $A, C \in \mathscr{T}$
(iv) $A \in \mathscr{T}$ if $B \in \mathscr{T}$ and (*) is pure.

Classes satisfying (i), (ii) and (iii) are called *torsion classes* (of abelian groups) and were first studied by Dickson [2], who classified those which contain only torsion groups and showed that the general classification problem reduces, essentially, to that for torsion classes determined (in the sense of § 2 below) by torsion-free groups. The torsion classes which are closed under taking subgroups (called *strongly-complete Serre classes*) can be described quite simply ([1], [2], [10]). A possible approach to the general problem is to investigate torsion classes closed under taking the subgroups corresponding to *proper classes* of monomorphisms as used in relative homological algebra (see for example [8], pp. 367 *et seqq.*), and herein lies the motivation for the present paper.

1. **Notation.** "Group" means "abelian group" throughout. $h(x)$ denotes the height of an element of a torsion-free group $\tau(x)$ its type and $\tau(X)$ the type of a rational group X. An *S-group*, where S is a set of primes, is a group whose elements have orders belonging to the multiplicative semigroup S^* generated by S. A group A is *p-divisible* for a prime p if $pA = A$ and *S-divisible* if p-divisible for each $p \in S$. $\mathscr{T}_0$, $\mathscr{F}_0$ are the classes of all torsion and torsion-free groups respectively. For a group A, A_t is the torsion subgroup, A_p its p-primary component. The direct sum (or discrete direct sum) of a set of groups $\{A_\mu \mid \mu \in M\}$ is denoted by $\sum A_\mu$, the direct product (or complete direct sum) by $\sum^* A_\mu$ and an element of either by (a_μ). $[A, B]$ is the group of homomorphisms from a group A to a group B. If a is an element of a torsion-free group A, $[a]$ denotes the cyclic subgroup it generates, $[a]_*$ the smallest pure subgroup containing it. Z is the group of integers, Q the (additive) group of rational numbers, $Z(p)$ the cyclic group of order p, $Z(p^\infty)$ the quasicyclic p-group. For

a set S of primes, $Q(S)$ is the subgroup $\{m/n \mid m \in Z, n \in S^*\}$ of Q and for a prime p, $Q(p) = \{m/p^n \mid m, n \in Z, n \geqq 0\}$. I_p is the group or ring of p-adic integers.

For unexplained terms see [4].

2. Torsion classes. We begin by listing some properties of torsion classes for later use.

For a class $\mathscr{C}$ of groups we write $T(\mathscr{C})$ for the torsion class determined by $\mathscr{C}$, i.e. the smallest torsion class $\mathscr{T}$ with $\mathscr{C} \subseteqq \mathscr{T}$ but if $\mathscr{C}$ has a single member C, $T(C)$ rather than $T(\{C\})$ will be used.

T1. *$A \in T(\mathscr{C})$ if and only if $[A, B] = 0$ whenever $[C, B] = 0$ for all $C \in \mathscr{C}$.* [3].

$T(\mathscr{C})$ is also the lower radical class determined by $\mathscr{C}$, in the sense of Kurosh [7]-Shul'geifer [9], so by the simplified version of the Kurosh construction which applies in an abelian category, we obtain

T2. *$A \in T(\mathscr{C})$ if and only if every nonzero homomorphic image B of A has a nonzero subgroup which is a homomorphic image of some $C \in \mathscr{C}$, i.e., $[C, B] \neq 0$.*

A torsion class $\mathscr{T}$ will be called a *t-torsion class* if it contains only torsion groups.

T3. *Let S_1, S_2 be disjoint sets of primes and let $\mathscr{T}$ be the class of all groups of the form $A_1 \oplus A_2$, where A_1 is an S_1-group and A_2 a divisible S_2-group. Then $\mathscr{T}$ is the t-torsion class*

$$T(\{Z(p) \mid p \in S_1\} \cup \{Z(p^\infty) \mid p \in S_2\}) .$$

Any t-torsion class is uniquely represented in this way. [2].

T4. *Let $\mathscr{T}$ be a torsion class and p a prime. Then either $Z(p) \in \mathscr{T}$ or every group in $\mathscr{T}$ is p-divisible* [2].

PROPOSITION 2.1. *If $\mathscr{T}$ is a torsion class containing a torsion-free group A, then $Z(p^\infty) \in \mathscr{T}$ for every prime p.*

Proof. If $Z(p) \in \mathscr{T}$, then $\mathscr{T}$ contains all p-groups (T3); if not, then A is p-divisible, so $\tau([a]_*) \geqq \tau(Q(p))$ for any nonzero $a \in A$. Thus $A/[a]$ has a subgroup and therefore a direct summand isomorphic to $Z(p^\infty)$, i.e. $Z(p^\infty)$ is a homomorphic image of A.

T5. *A torsion class $\mathscr{T}$ contains a group A if and only if A_t*

and $A/A_t \in \mathscr{T}$ [2].

T6. *Any torsion class* $\mathscr{T}$ *satisfies the equality*

$$\mathscr{T} = T([\mathscr{T} \cap \mathscr{T}_0] \cup [\mathscr{T} \cap \mathscr{F}_0]) .$$

[2].

T7. $T(Q(S))$ *is the class of S-divisible groups, for any set* S *of primes.* (Cf. [2], Proposition 4.1.)

3. A simplification of the problem. As a first step, we show that every torsion class closed under taking pure subgroups is either a t-torsion class or is determined by rational and torsion groups. A class of the latter kind will be called an *r.t.-torsion class.*

PROPOSITION 3.1. *All t-torsion classes are closed under taking pure subgroups.*

Proof. Let S_1, S_2 be disjoint sets of primes. If A_1 is an S_1-group and A_2 a divisible S_2-group, then clearly any pure subgroup of $A_1 \oplus A_2$ is the direct sum of an S_1-group and a divisible S_2-group.

THEOREM 3.2. *A torsion class* $\mathscr{T}$ *is closed under taking pure subgroups if and only if* $\mathscr{T} \cap \mathscr{F}_0$ *is.*

Proof. Let A' be a pure subgroup of $A \in \mathscr{T}$, and consider the induced diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & A'_t \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A'/A'_t \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A_t \arrow[r] & A \arrow[r] & A/A_t \arrow[r] & 0 \end{tikzcd}$$

with exact rows and columns, where g is defined by $g(a' + A'_t) = a' + A_t$. A'_t is pure in A' and hence in A. Therefore A'_t is pure in A_t so by Proposition 3.1, $A'_t \in \mathscr{T} \cap \mathscr{T}_0$. The kernel of g is $A' \cap A_t/A'_t = 0$. If, for some nonzero $n \in Z$, $a' \in A'$ and $a \in A$ we have $g(a' + A'_t) = n(a + A_t)$, then $m(a' - na) = 0$ for some nonzero $m \in Z$, i.e. $ma' = mna$. Since A' is pure in A, there exists $a'' \in A'$ with $ma' = mna''$. But then $g(a' + A'_t) = ng(a'' + A'_t)$, so that g is a pure monomorphism. Thus if $\mathscr{T} \cap \mathscr{F}_0$ is closed under taking pure subgroups, $A'/A'_t \in \mathscr{T} \cap \mathscr{F}_0$

so $A' \in \mathscr{T}$ and $\mathscr{T}$ is therefore closed under taking pure subgroups. The converse is obvious.

THEOREM 3.3. *If a torsion class $\mathscr{T}$ is closed under taking pure subgroups, then*

$$\mathscr{T} = T([\mathscr{T} \cap \mathscr{T}_0] \cup \overline{\mathscr{T}})$$

where $\overline{\mathscr{T}}$ is the class of rational groups in $\mathscr{T}$.

The proof uses the following lemmas:

LEMMA 3.4. *For $\mathscr{T}$ and $\overline{\mathscr{T}}$ as in Theorem 3.3, $\mathscr{T} \cap \mathscr{F}_0 = T(\overline{\mathscr{T}}) \cap \mathscr{F}_0$.*

Proof. Clearly $\mathscr{T} \cap \mathscr{F}_0 \supseteqq T(\overline{\mathscr{T}}) \cap \mathscr{F}_0$. Let A be any group in $\mathscr{T} \cap \mathscr{F}_0$. Then A is a homomorphic image of $\sum [a]_*$ where the sum extends over all $a \in A$ and each $[a]_* \in \mathscr{T}$, so $A \in T(\overline{\mathscr{T}})$.

LEMMA 3.5. *For any two classes $\mathscr{C}_1, \mathscr{C}_2$ of groups, $T(\mathscr{C}_1 \cup \mathscr{C}_2) = T(T[\mathscr{C}_1] \cup T[\mathscr{C}_2])$.*

To complete the proof of Theorem 3.3, we observe that

$$\begin{aligned}\mathscr{T} &= T([\mathscr{T} \cap \mathscr{T}_0] \cup [\mathscr{T} \cap \mathscr{F}_0]) = T([\mathscr{T} \cap \mathscr{T}_0] \cup [T(\overline{\mathscr{T}}) \cap \mathscr{F}_0]) \\ &\subseteqq T([\mathscr{T} \cap \mathscr{T}_0] \cup T(\overline{\mathscr{T}})) = T([\mathscr{T} \cap \mathscr{T}_0] \cup \overline{\mathscr{T}}) \subseteqq \mathscr{T} .\end{aligned}$$

We conclude this section by showing that not every r.t. torsion class is closed under taking pure subgroups.

PROPOSITION 3.6. *Let $\mathscr{T}$ be a torsion class closed under taking pure subgroups and Γ the set of types of rational groups in $\mathscr{T}$. If $\gamma, \delta \in \Gamma$, then $\gamma \cap \delta \in \Gamma$.*

Proof. Let X and Y be rational groups with $\tau(X) = \gamma$ and $\tau(Y) = \delta$. Then $X \oplus Y$ has elements and therefore pure rational subgroups of type $\gamma \cap \delta$.

Thus for example if p and q are distinct primes, $T(\{Q(p), Q(q)\})$ is not closed under taking pure subgroups since $\tau(Q(p)) \cap \tau(Q(q)) = \tau(Z)$ and $[Q(p), Z] = 0 = [Q(q), Z]$.

4. The main results.

In this section we obtain an explicit characterization of the torsion classes closed under taking pure subgroups.

LEMMA 4.1. *Let X be a rational group and $S = \{p \text{ prime} \mid X$ is* *p-divisible*$\}$. *Then* $I_p \in T(X)$ *whenever* $p \notin S$.

Proof. Let P be the set of primes distinct from p. Then $I_p \in T(Q(P))$(T7). Also, there is a short exact sequence

$$0 \longrightarrow X \longrightarrow Q(P) \longrightarrow \sum Z(q^\infty) \longrightarrow 0$$

where q ranges over $P - S$. Since $\sum Z(q^\infty) \in T(X)$(Proposition 2.1), it follows that $T(X)$ contains $Q(P)$ and hence I_p.

The main result can now be stated.

THEOREM 4.2. *A torsion class* $\mathscr{T}$ *is closed under taking pure subgroups if and only if either*
(i) $\mathscr{T}$ *is a t-torsion class*
or (ii) $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$, *where* P *and* S *are sets of primes with* $P \subseteqq S$.

The proof of Theorem 4.2 will be accomplished in several stages. We first prove

LEMMA 4.3. *Let* $\{X_\mu \mid \mu \in M\}$ *be a set of rational groups. Let* $A = \sum X_\mu$ *and* $S = \{p$ *prime* $\mid A$ *is p-divisible*$\}$. *Then* $T(\{X_\mu \mid \mu \in M\})$ *contains* $\sum {}^*A_i$, $i = 1, 2, 3, \cdots$, *where each* $A_i = A$.

Proof. Let $f\colon \sum {}^*A_i \rightarrow Y$ be a nonzero epimorphism. We show that $[X_\mu, Y] \neq 0$ for at least one value of μ.

If $Y_p \neq 0$ for some p, then since Y is S-divisible, so is Y_p. If $p \in S$, Y_p is therefore a direct sum of copies of $Z(p^\infty)$ so by Proposition 2.1, $Y_p \in T(X_\mu)$ for each μ and *a fortiori* $[X_\mu, Y] \neq 0$ for all μ. If $p \notin S$, then at least one X_μ is p-reduced, whence $[X_\mu, Y_p] \neq 0$.

If Y is torsion-free, there are two possibilities. If $f((a_i)) \neq 0$ for some (a_i) with almost all $a_i = 0$, then f induces a nonzero map from some A_i, and hence from some X_μ, into Y, while if $f((a_i)) = 0$ whenever $a_i = 0$ for almost all values of i, then f factorizes as

$$\begin{tikzcd} \sum^* A_i \arrow[r, "f"] \arrow[d] & Y \\ \sum^* A_i/\sum A_i \arrow[ur, dashed] & \end{tikzcd}$$

where the other maps are epimorphisms. $\sum^* A_i/\sum A_i$ is algebraically compact (see [6]), and also torsion-free, since $\sum A_i$ is a pure subgroup of $\sum^* A_i$. Thus $\sum^* A_i/\sum A_i$ is the direct sum of a divisible group and a (reduced) cotorsion group [5]; so, therefore, is Y, which being torsion-free is algebraically compact [5]. Since Y is S-divisible, it has the form $D \oplus \sum^* R_p$, $p \notin S$ where each R_p is *inter alia* a reduced I_p-module and D is divisible. If $D \neq 0$ then for each $\mu \in M$ there are monomorphisms $X_\mu \to Q \to D$. If $D = 0$, let $R_p \neq 0$. Then at least one X_μ is p-reduced, so by Lemma 4.1, $I_p \in T(X_\mu)$. Since there is an epimorphism (an I_p-epimorphism) from a direct sum of copies of I_p to R_p, we have $R_p \in T(I_p) \subseteq T(X_\mu)$, so $[X_\mu, R_p] \neq 0$ and the proof is complete.

The next step is to show when $T(\{X_\mu \mid \mu \in M\})$ is closed under taking pure subgroups.

LEMMA 4.4. *With the notation of Lemma* 4.3, *if* $T(\{X_\mu \mid \mu \in M\})$ *is closed under taking pure subgroups, it contains* $Q(S)$.

Proof. Let $p_1, p_2, p_3, \cdots$ be the natural enumeration of the primes, and let $J = \{i \mid p_i \notin S\}$. For each $j \in J$, choose $a_j \in A$ with $h_j(a_j) = 0$, where h_j denotes height at p_j. For example, let $a_j = (x_{j\mu})$ with $x_{j\mu} \in X_\mu$ satisfying the following conditions: (i) $x_{j\lambda} \neq 0$ for some $\lambda \in M$ for which X_λ is p_j-reduced; (ii) $h_j(x_{j\lambda}) = 0$; (iii) $x_{j\mu} = 0$ for $\mu \neq \lambda$. For a natural number $i \notin J$, let a_i be an arbitrary element of A, and regard the resulting (a_i) as an element of a group $\sum^* A_i$, $i = 1, 2, 3, \cdots$. Then $h((a_i)) = \bigcap_{i=1}^\infty h(a_i)$. In particular, $h_j((a_i)) = 0$. Therefore, since $\sum^* A_i$ is S-divisible, the height of (a_i) at a prime p is infinite if $p \in S$ and zero otherwise, i.e., $\tau((a_i)) = \tau(Q(S))$ and $\sum^* A_i$ has a pure subgroup isomorphic to $Q(S)$. By Lemma 4.3 and assumption, therefore, $Q(S) \in T(\{X_\mu \mid \mu \in M\})$.

Since each X_μ is S-divisible and $T(Q(S))$ is the class of all S-divisible groups (T7) we have

COROLLARY 4.5. *With the notation of Lemma* 4.3, *if* $T(\{X_\mu \mid \mu \in M\})$ *is closed under taking pure subgroups, it is the class of all S-divisible groups.*

Proof of Theorem 4.2. Let $\mathscr{T}$ be a torsion class closed under taking pure subgroups. If $\mathscr{T}$ is not a t-torsion class, let Γ be the set of types of rational groups in $\mathscr{T}$ and for each $\gamma \in \Gamma$ let X_γ be a rational group of type γ. Then

$$\mathscr{T} = T([\mathscr{T} \cap \mathscr{T}_0] \cup \{X_\gamma \mid \gamma \in \Gamma\}) \qquad \text{(Theorem 3.3)}$$

$$\text{and } \mathscr{T} \cap \mathscr{F}_0 = T(\{X_\gamma \mid \gamma \in \Gamma\}) \cap \mathscr{F}_0 \qquad \text{(Lemma 3.4).}$$

By Theorem 3.2, $T(\{X_\gamma \mid \gamma \in \Gamma\})$ is closed under taking pure subgroups and therefore, by Corollary 4.5, is the class of all S-divisible groups, where S is the set of all primes dividing $\sum X_\gamma$. Thus $\mathscr{T} =$

$T([\mathscr{T} \cap \mathscr{T}_0] \cup \{Q(S)\})$. Let $P = \{p \in S \mid Z(p) \in \mathscr{T}\}$. Since $T(Q(S)) \subseteqq \mathscr{T}$, $\mathscr{T}$ contains the groups $Z(p^\infty)$ for all primes p as well as $Z(p)$ for primes $p \notin S$. Thus by T3 and Lemma 3.5

$$\begin{aligned} \mathscr{T} &= T(\{Z(p) \mid p \notin S\} \cup \{Z(p) \mid p \in P\} \cup \{Z(p^\infty) \mid \text{all } p\} \cup \{Q(S)\}) \\ &= T(\{Z(p) \mid p \in P\} \cup \{Q(S)\}) . \end{aligned}$$

Conversely, that any class $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ with $P \subseteqq S$ is closed under taking pure subgroups follows from Theorem 3.2, Lemma 3.4 and the observation that $T(Q(S))$ is closed under taking pure subgroups. By Proposition 3.1, the proof is now complete.

Note that by T1, for a torsion class $\mathscr{T}$ which is not a t-torsion class, the representation $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ is unique. We conclude with a characterization of the groups in such a class:

PROPOSITION 4.6. *A group A belongs to $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ where P and S are sets of primes with $P \subseteqq S$, if and only if there is a short exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

where A' is a P-group and A'' is S-divisible.

Proof. Let $A \in \mathscr{T}$ and $A' = \sum A_p$ where the sum extends over all $p \in P$, $A'' = A/A'$. Then A''_t has no P-component and belongs to $\mathscr{T}$(T5) so therefore has divisible S-component. Thus A''_t is S-divisible. A''/A''_t is torsion-free and belongs to $\mathscr{T}$. If not S-divisible, it has a non-zero S-reduced torsion free homomorphic image B. But then $B \in \mathscr{T}$ and $[Q(S), B] = 0 = [Z(p), B]$ for each $p \in P$ and this contradicts T1, so A''/A''_t is S-divisible, whence A'' is also. The converse is obvious.

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