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INTEGRAL DOMAINS THAT ARE NOT EMBEDDABLE IN DIVISION RINGS

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A class of totally ordered rings V is constructed having the property $1 < \alpha \in V \Rightarrow 1/\alpha \in V$, but such that V cannot be embedded in any division ring.

1. Inverses in semigroup power series rings.

This note has only one objective—to construct the above class of counterexamples (see [6]).

NOTATION 1.1. Throughout Γ will be a totally ordered cancellative semigroup with identity e; R will denote any totally ordered division ring. If $\alpha: \Gamma \Rightarrow R$ is any function, then the *support* of α is the set $\operatorname{supp} \alpha = \{s \in \Gamma \mid \alpha(s) \neq 0\}$. The set $V = V(\Gamma, R)$ of all functions α such that $\operatorname{supp} \alpha$ satisfies the a.c.c. (ascending chain condition) form a totally ordered abelian group. If Γ is cancellative, then under the usual power series multiplication (see [3]), V is a totally ordered ring.

1.2. Any $1 < \alpha \in V$ with $\alpha(s) = 0$ for $s > e$ may be written as $\alpha = \alpha(e)(1 - \lambda)$, where $1 \leqq \alpha(e)$ and $\lambda = \Sigma\{\lambda(a)\, a \mid a < e\}$. It will be shown that

$$(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots = 1 + \textstyle\sum_s \sum' \lambda(a(1)\lambda(a(2)) \cdots \lambda(a(n))\,,$$

where the finite sum Σ' is over all integers and over all distinct n-tuples of Γ^n satisfying $s = a(1)a(2) \cdots a(n)$ with each $a(i) < e$; the sum Σ is over all $s < e$. To prove that $1/\alpha \in V$ it suffices to establish conditions (a) and (b) below.

(a) For each $s \in \Gamma$, there are only a finite number of n with $\lambda^n(s) \neq 0$;

(b) $\operatorname{supp}(1 - \lambda)^{-1}$ satisfies the a.c.c.

Assuming (a) and (b), the main theorem follows at once. By adjoining an identity as in [8; p. 158] to the semigroup in [2] a semigroup that actually satisfies the hypothesis in (ii) below can be constructed.

MAIN THEOREM 1.3. *If Γ is a totally ordered cancellative semigroup with identity e and R any totally ordered division ring, then the power series ring $V = V(\Gamma, R)$ has the following properties*:

(i) $1 < \alpha \in V$ and $\alpha(s) = 0$ for $s > e \Longrightarrow 1/\alpha \in V$.

(ii) *If in addition Γ cannot be embedded in a group, then V*

cannot be embedded in a division ring.

An already known result ([8; p. 135]) follows immediately from 1.3 (i).

COROLLARY 1.4. *If in addition Γ is a group, then $V(\Gamma, R)$ is a division ring.*

2. **Proof of the main theorem.** Assume 1.2 (a) or (b) fails. Then a lengthy but elementary argument shows there exists a doubly indexed matrix $\{a(i, j) \in \operatorname{supp}\lambda \mid 1 \leqq i < \infty; 1 \leqq j \leqq n(i)\}$ such that the products $u(i) = a(i, 1)a(i, 2) \cdots a(i, n(i))$ of the rows form an infinite properly ascending chain. Eventually a contradiction will be derived from this. Without loss of generality assume $\Gamma \leqq e$.

DEFINITION 2.1. For any totally ordered semigroup Γ with identity e and any element $a \in \Gamma$ with $a \leqq e$, define a semigroup by

$$\Gamma(a) = \{q \in \Gamma \mid \exists \text{ an integer } m > 0,\ q^m \leqq a\} .$$

LEMMA 2.2. *With Γ as above, for any $a(1), \cdots, a(m) \in \Gamma$ with each $a(j) \leqq e$, set $u = a(1)a(2) \cdots a(m)$ and define*

$$a^* = \min \{a(1), \cdots, a(m)\} .$$

Then $\Gamma(u) = \Gamma(a^*)$.

2.3. Consider a fixed subset $L \subseteqq \Gamma$ all of whose elements satisfy $L \leqq e$ and where L satisfies the a.c.c., e.g., $L = \operatorname{supp}\lambda < e$. Consider an array of elements $A = \|a(i, j)\|$ with $\{a(i, j) \mid 1 \leqq i < \infty, 1 \leqq j \leqq n(i)\} \subseteqq L$, where repetitions in the $a(i, j)$ are allowed. Assume all $n(i) \geqq 2$. Define $u(i) = u(i, A)$ by

$$u(i) = u(i, A) = a(i, 1)a(i, 2) \cdots a(i, n(i)) .$$

Let $\mathscr{K}$ be the set of all such $A = \|a(i, j)\|$ for which $u(1) < u(2) < \cdots < u(i) < \cdots$ is strictly ascending at each i. With each member $A = \|a(i, j)\| \in \mathscr{K}$, we next associate three objects

$$\{a(i)^* \mid 1 \leqq i < \infty\},\ m = m(A), \text{ and } G = G(A) .$$

Define $a(i)^* \equiv \min \{a(i, j) \mid 1 \leqq j \leqq n(i)\}$. Note that $u(1) < u(2) < \cdots$ implies that $\Gamma(a(1)^*) \subseteqq \Gamma(a(2)^*) \subseteqq \Gamma(a(i)^*) \subseteqq \cdots$. Thus since L satisfies the a.c.c., there is a unique smallest integer $m \equiv m(A)$ such that the semigroups $G \equiv \Gamma(a(m)^*) = \Gamma(a(m + 1)^*) = \cdots$ are all equal. The following schematic diagram of all these quantities may be helpful.

$$\begin{array}{ll}
\Gamma(a(1)^*) = \Gamma(u(1)) & u(1) = a(1,1)a(1,2)\cdots a(1)^*\cdots a(1,n(1)) \\
\qquad\qquad\quad \cap\!\!\parallel & \\
\Gamma(a(2)^*) = \Gamma(u(2)) & u(2) = a(2,1)a(2,2)\cdots a(2)^*\cdots a(2,n(2)) \\
\qquad\qquad\quad \cap\!\!\parallel & \\
\Gamma(a(m)^*) = \Gamma(u(m)) & u(m) = a(m,1)a(m,2)\cdots a(m)^*\cdots a(m,n(m)) \\
\qquad\qquad\quad \parallel & \\
\qquad G = \Gamma(u(m+1))\,. &
\end{array}$$

2.4. Among the elements of $\mathscr{K}$, let $\mathscr{N}\subset\mathscr{K}$ be all those $A = \|a(i,j)\|$ such that this associated $G = G(A)$ is as big as possible and call this particular $G \equiv M$. If $\mathscr{K}\neq\varnothing$, also $\mathscr{N}\neq\varnothing$. Define $\bar{a} = \max\{a(m)^* \mid A\in\mathscr{K},\, m = m(A)\}$. Pick and element $B = \|b(i,j)\|\in\mathscr{N}$. Then by our choice of M, $\Gamma(\bar{a}) = M$. Thus $M = G(B) = \Gamma(b(i)^*) = \Gamma(b(i,j)) = \Gamma(u(i)) = \Gamma(\bar{a})$ for $i \geqq m(B) \equiv m$. Finally, with each element B of $\mathscr{N}$, we associate an integer $r = r(B)$. Since $\bar{a}\in\Gamma(u(m))$, there is a unique smallest integer $r \equiv r(B) \geqq 1$ such that $\bar{a}^r \leqq u(m) < \bar{a}^{r-1}$.

2.5. By omitting some of the rows of B and renumbering the remaining ones, it may be assumed as a consequence of the a.c.c. without loss of generality that $m = 1$, and also that $b(1)^* \geqq b(2)^* \geqq \cdots$ is not ascending. Each $u(i)$ is of one of the following three forms:

$$u(i) = q(i)b(i)^*\,, \tag{1}$$

$$u(i) = b(i)^*w(i)\,, \tag{2}$$

$$u(i) = q(i)b(i)^*w(i)\,, \tag{3}$$

where the $q(i)$, $w(i)$ are certain products of the $b(i,j)$. If there are an infinite number of $u(i)$ of the forms (1) or (2), then since

$$u(i+1) = q(i+1)b(i+1)^* > u(i) = q(i)b(i)^*,\ b(i+1)^* \leqq b(i)^* \\ \Longrightarrow q(i+1) > q(i)\ ;$$

it follows (after omitting some rows and renumbering) that there is a properly infinite ascending chain:

Case 1. $q(1) < q(2) < \cdots$;

Case 2. $w(1) < w(2) < \cdots$.

If neither Case 1 nor Case 2 applies, then

$$u(i+1) = q(i+1)b(i+1)^*w(i+1) > q(i)b(i)^*w(i) \\ \text{and } b(i+1)^* \leqq b(i)^*$$

implies that one of the inequalities $q(i+1) > q(i)$ or $w(i+1) > w(i)$

must necessarily hold. It is asserted that there is a subsequence $\{i(k) \mid k = 1, 2, \cdots\}$ such that

$$\text{\textit{Case} 3. either (a): } q(i(1)) < q(i(2)) < \cdots$$
$$\text{or (b): } w(i(1)) < w(i(2)) < \cdots .$$

For if not, then the a.c.c. must hold in both the sets $\{q(i)\}$ and $\{w(i)\}$. Then by omitting some rows and renumbering the remaining ones it may be assumed that we have an element B in $\mathscr{N}$ with $q(1) \geqq q(2) \geqq \cdots$ and $w(1) \geqq w(2) \geqq \cdots$. However, then

$$q(1)b(1)^* w(1) \geqq q(2)b(2)^* w(2) \geqq \cdots$$

gives a contradiction.

2.6. We may assume $q(1) < q(2) < \cdots$ or $w(1) < w(2) \cdots$ are properly ascending, depending on which of the Cases 1, 2, 3(a) or 3(b) is applicable. Set $t = r(B)$, so that $\bar{a}^t \leqq u(m) = u(1) \leqq u(i)$.

2.7. It is next shown that either $q(i) \geqq \bar{a}^{t-1}$ or $w(i) \geqq \bar{a}^{t-1}$ holds for all i. Suppose that the following holds.

$$\begin{array}{llll} \textit{Case } 1. & q(1)b(1)^* & < q(2)b(2)^* & < \cdots; \\ & q(1) & < q(2) & < \cdots; \\ & b(1)^* \geqq & b(2)^* & \geqq \cdots . \end{array}$$

Then $\bar{a}^t \leqq u(1) \leqq u(i) = q(i)b(i)^*$, and $\bar{a} \geqq b(i)^*$ implies that

$$\bar{a}^{t-1} \leqq q(1) \leqq q(i) .$$

(For if $\bar{a}^{t-1} > q(i)$, then $\bar{a} \geqq b(i)^*$ implies that $\bar{a}^t > q(i)b(i)^*$.) (If $t = 1$, then $\bar{a}^0 = e$.) Similarly, in Case 2 also $\bar{a}^{t-1} \leqq w(1) \leqq w(i)$.

Only Case 3(b) will be proved, since 3(a) is entirely parallel.

$$\begin{array}{llll} \textit{Case } 3(b). & q(1)b(1)^* w(1) & < q(2)b(2)^* w(2) & < \cdots; \\ & w(1) & < w(2) & < \cdots; \\ & b(1)^* \geqq & b(2)^* & \geqq \cdots . \end{array}$$

Then again $\bar{a}^t \leqq u(1) \leqq u(i) = q(i)b(i)^* w(i)$ and $\bar{a} \geqq b(i)^* \geqq q(i)b(i)^*$ imply that $\bar{a}^{t-1} \leqq w(1) \leqq w(i)$. (Otherwise, if $\bar{a}^{t-1} > w(i)$, then $\bar{a}^t > q(i)b(i)^* w(i)$.)

The basic idea motivating the proof is that for $B \in \mathscr{N}$, a new $C \in \mathscr{N}$ can be constructed with $r(C) \leqq r(B) - 1$.

2.8. Thus either $q(1) < q(2) < \cdots$ and all $q(i) \geqq \bar{a}^{t-1}$; or $w(1) < w(2) < \cdots$ and all $w(i) \geqq \bar{a}^{t-1}$. Assume the latter. Let

$$C = ||c(i, j)|| \in \mathscr{K}$$

be defined by taking as its i-th row all the $b(i, j)$ appearing in $w(i)$. (In view of $w(1) < w(2) < \cdots$, there does not exist an infinite number of rows of C containing only one element. By omitting a finite number of rows it may be assumed that all rows of C contain two or more elements of L.) Define $c(i)^* \equiv \inf \{c(i, j) \mid j \geqq 1\}$. Since $b(i)^* \leqq c(i)^* \leqq \bar{a}$, it follows that

$$\boldsymbol{M} = \Gamma(b(i)^*) \subseteqq \Gamma(c(i)^* \subseteqq \Gamma(\bar{a}) = \boldsymbol{M} .$$

Consequently, $\boldsymbol{G}(C) = \boldsymbol{M}$ and $C \in \mathscr{N}$. Since $w(1) \geqq \bar{a}^{t-1}$, $r(C) \leqq t - 1$. By repetition of this process, we may reduce the r to one so that finally $\bar{a}^r = \bar{a} \leqq w(1) < w(2) \cdots$. Since all $c(i, j) \in L$ satisfy $c(i, j) \leqq e$ and since $w(i)$ is a product of these, it follows that $\bar{a} \geqq c(i)^* \geqq w(i)$. Thus $\bar{a} = w(1) = w(2) = \cdots$ gives a contradiction. Thus $\mathscr{K} = \varnothing$ and the main theorem has been proved.

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