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THE FOUR-PERSON CONSTANT-SUM GAMES: DISCRIMINATORY SOLUTIONS ON THE MAIN DIAGONAL

GUILLERMO OWEN

It is known that every four-person constant-sum game has discriminatory solutions. In this paper, we consider the games on the "main diagonal" which are symmetric in the first three players, and look for solutions which discriminate the fourth player, i.e., give him a constant amount. The seven types of solutions are catalogued, and necessary and sufficient conditions are found for the solution of the 3-person game to expand to a solution of the 4-person game. Finally, this paper determines the amounts which the fourth, discriminated player is allowed to receive in order that a solution of each of the seven types exist.

The four-person constant-sum games (in (0, 1)-normalization) can, as is well known, be represented by a unit cube $0 \leqq v_i \leqq 1$, where $v_i = 1 - v(\{i, 4\})$. We consider here games on the "main diagonal," $v_1 = v_2 = v_3 = U$. These are, of course, symmetric in $\{1, 2, 3\}$. We look for solutions which discriminate the remaining player, 4.

DEFINITION. Let v be an n-person game, let S be a coalition, and let q be a number. Then by $\bar{v}_{s,q}$ we mean the game with player set S, defined by

$$\bar{v}(T) = \begin{cases} v(T) & \text{if } T \subset S,\ T \neq S \\ q & \text{if } T = S\,. \end{cases}$$

The following theorems are given without proof (see [2]):

THEOREM 1. *Let v be an n-person game, and let V be a solution which discriminates the members of $N - S$, giving them the amounts α_j. Then the S-components of the elements of V form a solution to the game $\bar{v}_{s,q}$, where*

$$q = v(N) - \sum_{N-S} \alpha_j\,.$$

THEOREM 2. *Let v be an n-person game, let α be an $(N - S)$-vector with $\alpha_j \geqq v(\{j\})$, and let*

$$q = v(N) - \sum_{N-S} \alpha_j\,.$$

Let V^* be a solution to $\bar{v}_{s,q}$, and let V be obtained from V^* by adjoining the components (α_j) to the elements of V^*. Then, a necessary and sufficient condition for the set V to dominate all imputations x, with

$$(1) \qquad \sum_{N-S} (x_j - \alpha_j) > 0$$

is that either $v(S) \geqq q$ or the core of the game $\bar{v}_{s,q}$ have no interior points. Moreover, V is always internally stable.

Thanks to these theorems, the question of whether a solution of $\bar{v}_{s,q}$ expands to one of v, reduces to whether imputations outside of V, other than these satisfying (1), are dominated by V.

From the above, we know that a discriminatory solution to a 4-person game must have the form of a solution to a 3-person game. Now, these have been catalogued for us (see, e.g., [4]). For the games on the main diagonal, the 3-person games $\bar{v}_{s,q}$ where $S = \{1, 2, 3\}$, are symmetric, and there will be seven types of solution, types I through VII shown in Figures 1 through 7 respectively. Since the game v is constant-sum, we will always have $v(\{1, 2, 3\}) \geqq q$. Moreover, with only one player in $N - S$, we need only worry about imputations x with $x_4 > \alpha_4$. If $x_4 = \alpha_4$, then either $x \in V$, or $x \notin V$ and so $(x_1, x_2, x_3) \notin V^*$. Hence there is $y \in V$ such that (y_1, y_2, y_3) dominates (x_1, x_2, x_3) in $(\bar{v})$ and so $y > x$ (in v).

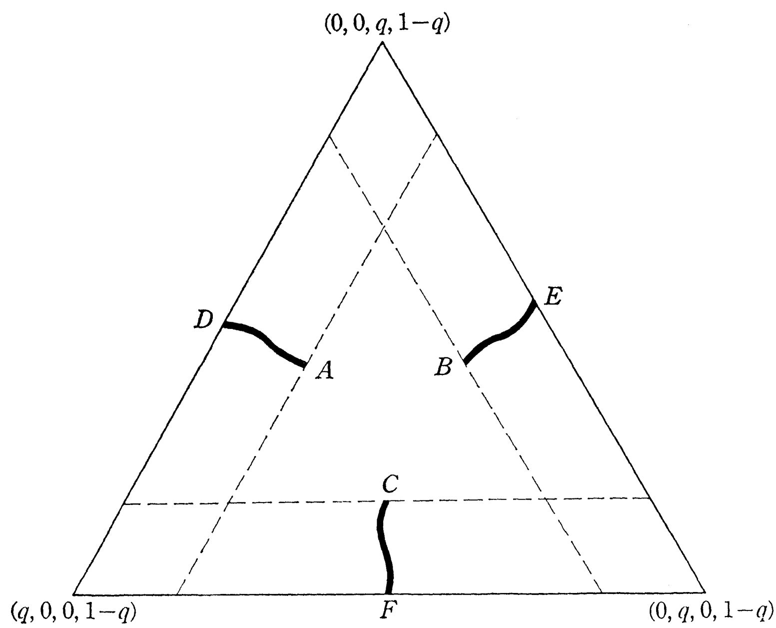

FIGURE 1

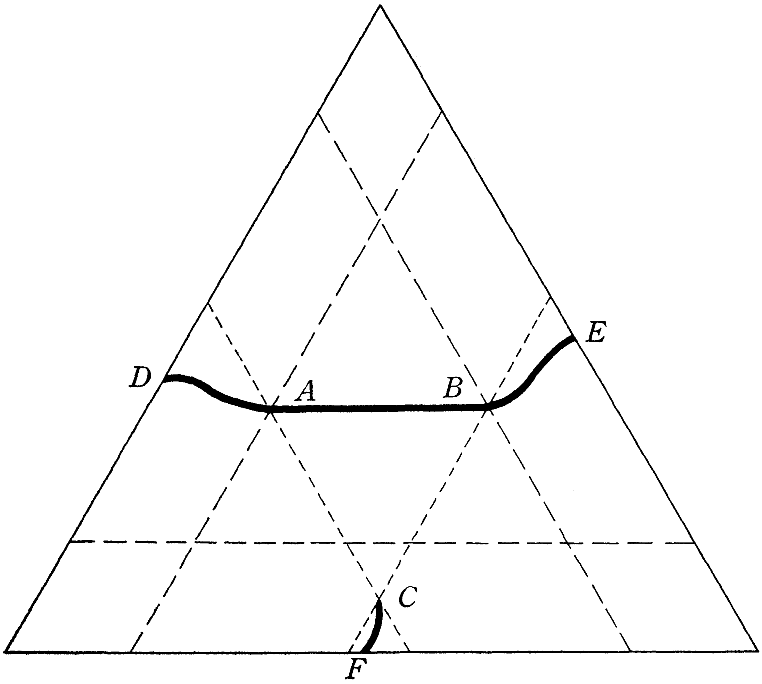

FIGURE 2

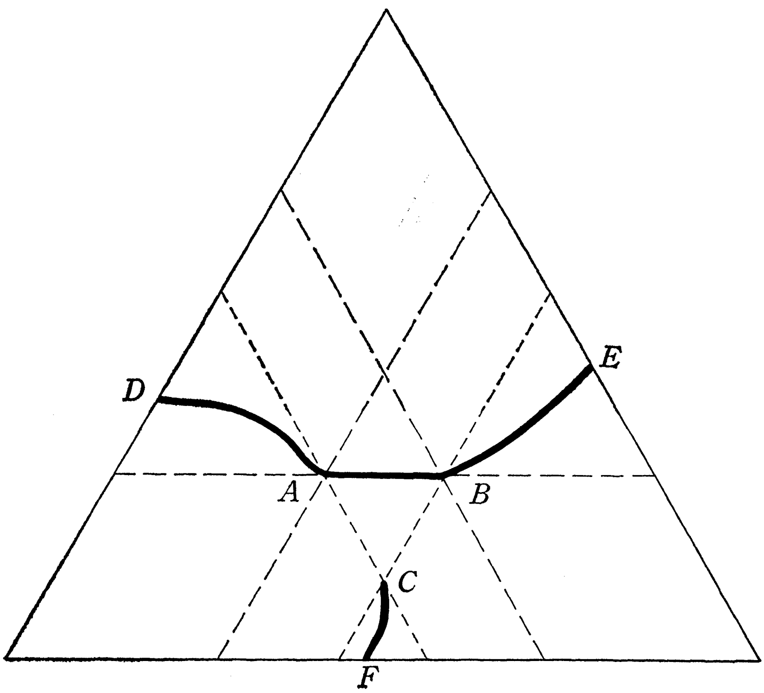

FIGURE 3

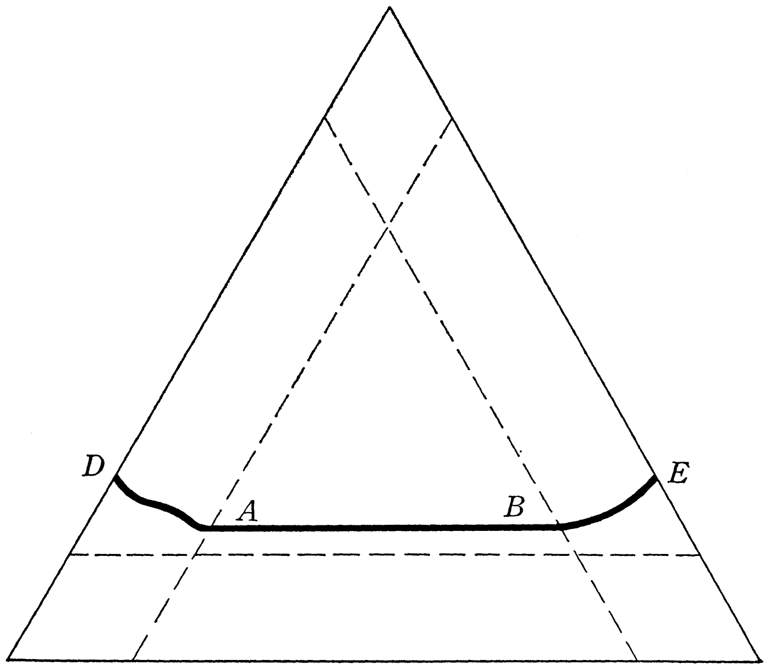

FIGURE 4

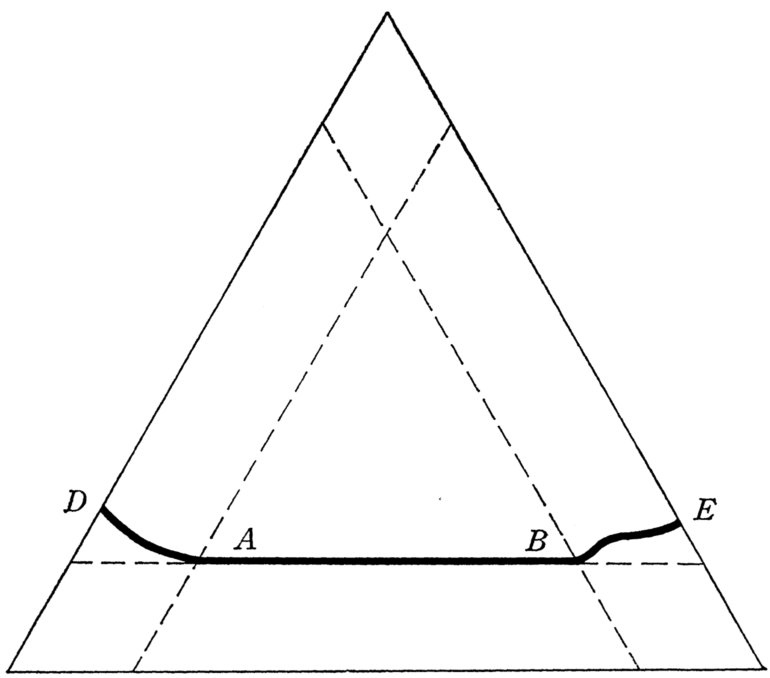

FIGURE 5

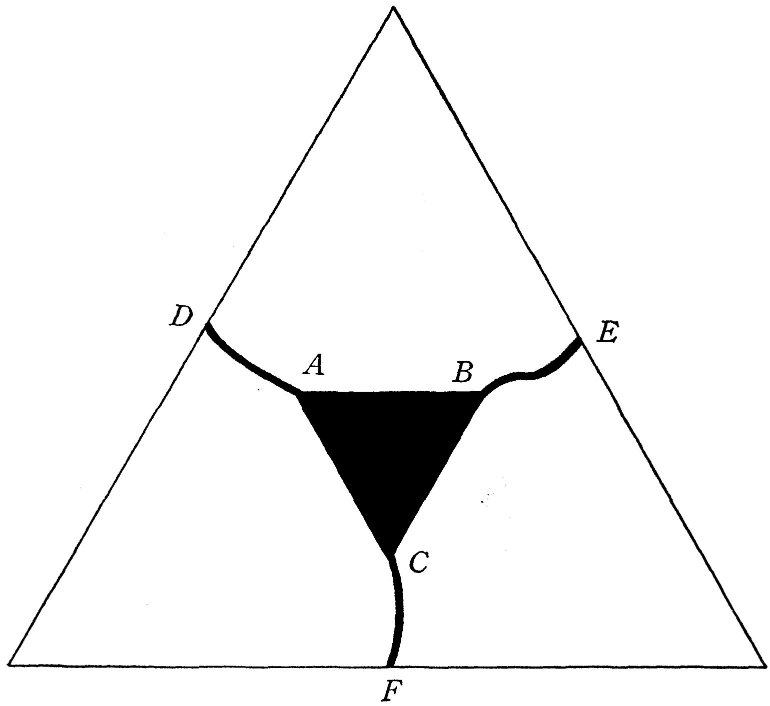

FIGURE 6

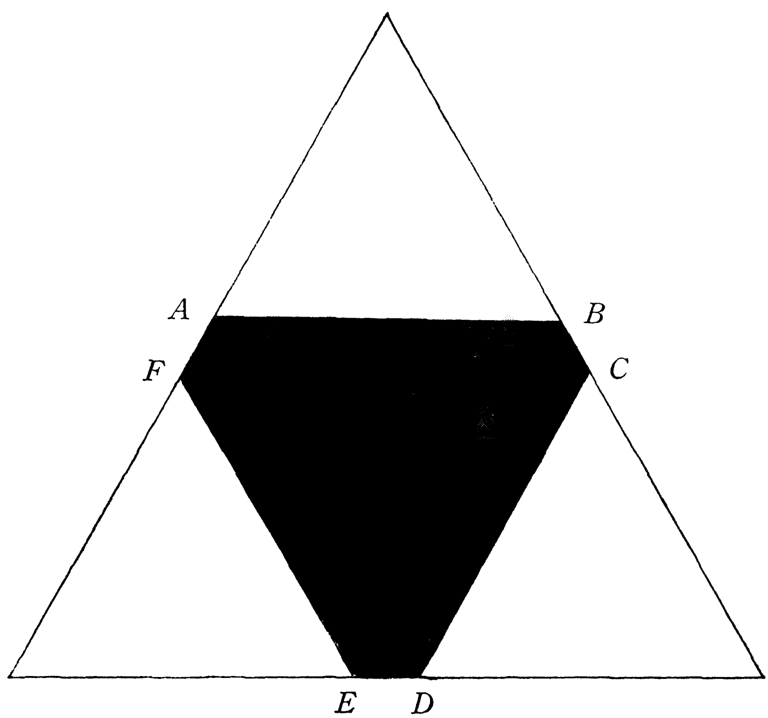

FIGURE 7

Note finally that we have $\alpha_4 = 1 - q$. We will use $1 - q$ throughout, rather than α_4.

2. Description of the several types. We describe now the seven types of solution.

Type I (see Figure 1) consists of the three points A, B, C (the mid-points of the inner triangle formed by the lines $x_i + x_j = U$) together with three "bargaining curves" AD, BE, and CF, arbitrary except for the proviso that, on any one of these curves, the shares of the two players who are receiving more than the third must increase or decrease simultaneously (see [4]).

The points A, B, C have the components:

$$A = (U/2, q - U, U/2, 1 - q) \tag{1}$$

$$B = (q - U, U/2, U/2, 1 - q) \tag{2}$$

$$C = (U/2, U/2, q - U, 1 - q) \tag{3}$$

while D, E, F will be

$$D = (d_1, 0, d_3, 1 - q) \tag{4}$$

$$E = (0, e_2, e_3, 1 - q) \tag{5}$$

$$F = (f_1, f_2, 0, 1 - q) \,. \tag{6}$$

Type II (See Figure 2) consists of the straight line AB, plus three bargaining curves AD, BE, CF. The straight line is parallel to $x_3 = 0$, less than half-way up the inner triangle, whereas the point C is the intersection of the lines through A and B, parallel to $x_1 = 0$ and $x_2 = 0$, respectively. The bargaining curves are arbitrary, except for the monotonicity conditions described above.

Now A, B, C are given by

$$A = (U - k, q - U, k, 1 - q) \tag{7}$$

$$B = (q - U, U - k, k, 1 - q) \tag{8}$$

$$C = (U - k, U - k, q - 2U + 2k, 1 - q) \tag{9}$$

where k is any number satisfying

$$q - U < k < U/2 \tag{10}$$

$$k \geqq U - q/2 \,. \tag{11}$$

Type III is very similar to type II, the sole difference lying in the fact that line AB is now the edge of the inner triangle, i.e., $k = q - U$. Thus

(12) $$A = (2U - q, q - U, q - U, 1 - q) ,$$

(13) $$B = (q - U, 2U - q, q - U, 1 - q) ,$$

(14) $$C = (2U - q, 2U - q, 3q - 4U, 1 - q) .$$

Type IV is quite similar to type II. The difference, here, lies in the fact that the lines through A and B, parallel to $x_1 = 0$ and $x_2 = 0$, intersect outside the triangle. Thus the point C disappears, as does, indeed the whole curve CF. We will have

(15) $$A = (U - k, q - U, k, 1 - q) ,$$

(16) $$B = (q - U, U - k, k, 1 - q) ,$$

with

(17) $$q - U < k < U/2 ,$$

(18) $$k < U - q/2 .$$

Type V is quite similar to type IV (its relation to type IV is the same as that of III to II). The points A and B are:

(19) $$A = (2U - q, q - U, q - U, 1 - q) ,$$

(20) $$B = (q - U, 2U - q, q - U, 1 - q) .$$

Type VI is considerably different. It consists of the triangle ABC (see Figure 6) which is the core of the three-person game $\bar{v}$, plus the three bargaining curves AD, BE, CF. The points A, B, C have coordinates

(21) $$A = (q - U, 2U - q, q - U, 1 - q) ,$$

(22) $$B = (2U - q, q - U, q - U, 1 - q) ,$$

(23) $$C = (q - U, q - U, 2U - q, 1 - q) .$$

Finally, type VII is the only type of solution which does not contain a bargaining curve (see Figure 7). It is the core of the three-person game, the hexagon with vertices

(24) $$A = (u, 0, q - U, 1 - q) ,$$

(25) $$B = (0, U, q - U, 1 - q) ,$$

(26) $$C = (0, q - U, U, 1 - q) ,$$

(27) $$D = (U, q - U, 0, 1 - q) ,$$

(28) $$E = (q - U, U, 0, 1 - q) ,$$

$$F = (q - U, 0, U, 1 - q) . \tag{29}$$

3. Domination of imputations. We now treat the question of consistency. Since internal consistency is trivial, we need only worry about external consistency. In this section we look for conditions that an imputation be undominated by the set V. As discussed above, we need only consider imputations with $x_4 < 1 - q$.

Suppose then, that $x_4 < 1 - q$. We wish to know whether there is some $y \in V$ with $y \succ x$. Now this domination can only be through a 2-person or 3-person coalition. It cannot be through $\{1, 2, 3\}$, as we know that, if $y \in V$ then $y_4 > x_4$, and so $y_1 + y_2 + y_3 < x_1 + x_2 + x_3$. It might be through a 2-person subcoalition of $\{1, 2, 3\}$, but if so, we can always add player 4 to this coalition, since the 3-person coalitions are all winning, and we have $y_4 > x_4$. Thus domination may be assumed to be through a 2- or 3-person coalition which includes player 4.

Suppose, then, that x, with $x_4 < 1 - q$, is dominated by some $y \in V$ through a 2-person coalition, say $\{1,4\}$. This means

$$y_1 > x_1 , \tag{1}$$

$$y_4 > x_4 , \tag{2}$$

$$y_1 + y_4 \leqq v(\{1,4\}) = 1 - U . \tag{3}$$

Clearly, condition (2) is satisfied by all $y \in V$, as $y_4 = 1 - q > x_4$. This means, moreover, that (3) reduces to

$$y_1 \leqq 1 - U - (1 - q) = q - U . \tag{4}$$

Thus the question of whether x is dominated through $\{1,4\}$ by some $y \in V$ reduces to whether there exists $y \in V$ satisfying (1) and (4). It becomes natural to look for that $y \in V$ which maximizes y_1, subject to condition (4). Then x will be dominated through $\{1,4\}$ by this point, if and only if $x_1 < y_1$. If x_1 is greater than this constrained maximum of y_1, then no $y \in V$ can dominate x through $\{1,4\}$.

Looking at the several types of solution, we see that, in each type, there is some point with $y_1 = q - U$. (This is point B for types I through V, point A for type VI, and point E for type VII.) We conclude that, for each type, a necessary and sufficient condition for x to be undominated by V through $\{1,4\}$ is

$$x_1 \geqq q - U .$$

Consider, next, domination through $\{2,4\}$. The situation here is exactly the same, and the condition for nondomination is

$$x_2 \geqq q - U .$$

We go on to domination through {3,4}. In this case, the symmetry of the situation is lost because there are two types (II and IV) with no point y satisfying $y_3 = q - U$. In type II, the critical point for {3,4} domination is C, with $y_3 = q - 2U + 2k$, while in type IV, there is no domination through {3,4}.

We consider, now, domination through the 3-person coalitions $\{i, j, 4\}$. We need worry about this domination only in case x is undominated through $\{i, 4\}$ and $\{j, 4\}$.

Take, for example, the coalition {1, 2, 4}. We know $y \succ x$ through {1, 2, 4} if and only if

$$y_1 > x_1 , \tag{5}$$

$$y_2 > x_2 , \tag{6}$$

$$y_4 > x_4 . \tag{7}$$

Condition (7) will hold automatically. Conditions (5) and (6) must hold simultaneously. But, if $x_1 < q - U$ or $x_2 < q - U$, x will be dominated by {1,4} and {2,4}. As we are not worried about such x, we may assume $x_i \geqq q - U$ for $i = 1, 2$. In this context, conditions (5) and (6) imply that $y_i > q - U$ for $i = 1, 2$. Thus we must look for points which maximize y_1 and y_2, subject to the constraint that both be greater than $q - U$.

Now, it happens that, for types I, II, III, and VI, point F maximizes both y_1 and y_2, subject to $y_1 > q - U$, $y_2 > q - U$. We conclude that, for these four types, the necessary and sufficient condition for nondomination through {1, 2, 4}, assuming x is not dominated through {1,4} or {2,4}, is:

$$\text{Either } x_1 \geqq f_1 \text{ or } x_2 \geqq f_2 .$$

For type IV and V solutions, the situation is slightly different. Here, there is no point which maximizes both y_1 and y_2; rather we find that the line segment AB maximizes the sum $y_1 + y_2$; it satisfies $y_1 + y_2 = q - k$ for type IV; $y_1 + y_2 = U$ for type V. What is more, the line AB will contain all points which satisfy this equation as well as $y_i \geqq q - U$ for $i = 1, 2$. Thus, if $x_1 + x_2 < q - k$, with $x_1 \geqq q - U$, $x_2 \geqq q - U$, we can find some y on AB with $y_1 > x_1$, $y_2 > x_2$. For type IV, the necessary and sufficient condition for nondomination through {1, 2, 4} (assuming nondomination through {1,4} and {2,4}) is thus

$$x_1 + x_2 \geqq q - k$$

while, for type V, it is

$$x_1 + x_2 \geqq U .$$

For type VII, the situation is quite simple: there is no $y \in V$ with $y_i > q - U$, $i = 1, 2$. Thus we need not worry about domination through {1, 2, 4}.

We go now to domination through {1, 3, 4}. For types I, VI, and VII, symmetry tells us that the results are similar to those for {1, 2, 4}. The critical point (for I and VI) is D.

For type II, lack of symmetry complicates the situation slightly, but we find that D is once again critical, as it maximizes y_1, and y_3, subject to $y_1 > q - U$, $y_3 > q - 2U + 2k$. This analysis is valid, with minor variations, for types III, IV, and V as well. Thus, for types I through VI, the condition

$$\textit{Either } x_1 \geqq d_1 \text{ or } x_3 \geqq d_3$$

is both necessary and sufficient for {1, 3, 4} nondomination, assuming no domination through {1,4} or {3,4}. For type VII, such domination is unimportant.

For {2, 3, 4} domination, symmetry makes the analysis exactly similar to that for {1, 3, 4}.

We conclude this section by giving a list of conditions for nondomination of x.

TABLE 1.

Types	I, III, VI	II	IV	V	VII
Coalitions {1,4}	$x_1 \geqq q - U$	$x_1 \geqq q - U$	$x_1 \geqq q - U$	$x_1 \geqq q - U$	$x_1 \geqq q - U$
{2,4}	$x_2 \geqq q - U$	$x_2 \geqq q - U$	$x_2 \geqq q - U$	$x_2 \geqq q - U$	$x_2 \geqq q - U$
{3,4}	$x_3 \geqq q - U$	$x_3 \geqq q - 2U + 2k$		$x_3 \geqq q - U$	$x_3 \geqq q - U$
{1,2,4}	$x_1 \geqq f_1$ or $x_2 \geqq f_2$	$x_1 \geqq f_1$ or $x_2 \geqq f_2$	$x_1 + x_2 \geqq q - k$	$x_1 + x_2 \geqq U$	
{1,3,4}	$x_1 \geqq d_1$ or $x_3 \geqq d_3$	$x_1 \geqq d_1$ or $x_3 \geqq d_3$	$x_1 \geqq d_1$ or $x_3 \geqq d_3$	$x_1 \geqq d_1$ or $x_3 \geqq d_3$	
{2,3,4}	$x_2 \geqq e_2$ or $x_3 \geqq e_3$	$x_2 \geqq e_2$ or $x_3 \geqq e_3$	$x_2 \geqq e_2$ or $x_3 \geqq e_3$	$x_2 \geqq e_2$ or $x_3 \geqq e_3$	

4. External stability. We consider now the question of whether a set V, of one of the types described, is really a solution. The condition for this is quite simply expressed. There must be no x which satisfies all of the conditions in the table above. More precisely, the nondomination conditions must be inconsistent with the conditions $x_i \geqq 0$, $\sum x_i = 1$.

Consider, thus, types I, III, and VI. The nondomination conditions consist of three single conditions and three pairs of alternatives. This means that there are eight sets of six conditions each. Each *of these eight sets is sufficient for nondomination, while one is necessary.*

If we choose the first condition from each pair of alternatives, we have

$$x_1 \geqq q - U\,, \tag{1}$$

$$x_2 \geqq q - U\,, \tag{2}$$

$$x_3 \geqq q - U\,, \tag{3}$$

$$x_1 \geqq f_1\,, \tag{4}$$

$$x_1 \geqq d_1\,, \tag{5}$$

$$x_2 \geqq e_2\,. \tag{6}$$

Now, f_1 and d_1 are both greater than $q - U$, as is e_2. Thus the conditions (1)–(6) reduce to four.

$$x_3 \geqq q - U$$
$$x_1 \geqq f_1$$
$$x_1 \geqq d_1$$
$$x_2 \geqq e_2\,.$$

We introduce the notation

$$g_1 = \text{Max}\,\{d_1, f_1\}$$
$$g_2 = \text{Max}\,\{e_2, f_2\}$$
$$g_3 = \text{Max}\,\{d_3, e_3\}$$

and conditions (1)–(6) reduce to

$$x_1 \geqq g_1$$
$$x_2 \geqq e_2$$
$$x_3 \geqq q - U\,.$$

For V to be an imputation, this must be inconsistent with the natural constraints $x_i \geqq 0$, $\sum x_i = 1$. But this happens if, and only if,

$$g_1 + e_2 + q - U > 1\,.$$

In a similar manner, each of the seven other sets of conditions will reduce to three conditions, which will be inconsistent with the natural constraints if a certain strict inequality holds. We have then:

THEOREM 3. *A necessary and sufficient condition for a set V, of types* I, III, *or* VI, *with $q < 1$, to be a solution, is that*

$$g_1 + e_2 + q - U > 1 \,, \tag{7}$$

$$d_1 + g_2 + q - U > 1 \,, \tag{8}$$

$$g_1 + e_3 + q - U > 1 \,, \tag{9}$$

$$g_2 + d_3 + q - U > 1 \,, \tag{10}$$

$$f_1 + g_3 + q - U > 1 \,, \tag{11}$$

$$f_2 + g_3 + q - U > 1 \,, \tag{12}$$

$$d_1 + f_2 + e_3 > 1 \,, \tag{13}$$

$$f_1 + e_2 + d_3 > 1 \,. \tag{14}$$

The other types of solutions can be treated similarly. For type II, we have

THEOREM 4. *A set of type* II, *with $q < 1$, will be a solution if and only if*:

$$g_1 + e_2 + q - 2U + 2k > 1 \,, \tag{15}$$

$$d_1 + g_2 + q - 2U + 2k > 1 \,, \tag{16}$$

$$g_1 + e_3 + q - U > 1 \,, \tag{17}$$

$$g_2 + d_3 + q - U > 1 \,, \tag{18}$$

$$f_1 + g_3 + q - U > 1 \,, \tag{19}$$

$$f_2 + g_3 + q - U > 1 \,, \tag{20}$$

$$d_1 + f_2 + e_3 > 1 \,, \tag{21}$$

$$f_1 + e_2 + d_3 > 1 \,. \tag{22}$$

For type IV, the situation is somewhat different. There are two pairs of alternatives, and hence four possibilities. The first possibility is

$$x_1 \geqq q - U \,, \tag{23}$$

$$x_2 \geqq q - U \,, \tag{24}$$

$$x_1 + x_2 \geqq q - k \tag{25}$$

$$x_1 \geqq d_1 \tag{26}$$

$$x_2 \geqq e_2 \,. \tag{27}$$

Now, $d_1 \geqq q - U$ and $e_2 \geqq q - U$. Moreover, we know that $q - 2U + 2k < 0$, as otherwise the solution would be of type II (i.e., the point C would be an imputation). But this means that $d_1 + e_2 \geqq q - 2k$, and as $k > 0$, $d_1 + e_2 \geqq q - k$. The five conditions (23)–(27) thus reduce to two:

$$x_1 \geqq d_1$$

$$x_2 \geqq e_2$$

and the condition for inconsistency is $d_1 + e_2 > 1$. We treat the other conditions similarly, to obtain

THEOREM 5. *A set of type* IV, *with* $q < 1$, *will be a solution if and only if*

$$d_1 + e_2 > 1 \,, \tag{28}$$

$$d_1 + e_3 + q - U > 1 \,, \tag{29}$$

$$e_2 + d_3 + q - U > 1 \,, \tag{30}$$

$$g_3 + q - k > 1 \,. \tag{31}$$

A somewhat similar treatment for type V gives us

THEOREM 6. *A set of type* V, *with* $q < 1$, *will be a solution if and only if*

$$d_1 + e_2 + q - U > 1 \,, \tag{32}$$

$$d_1 + e_3 + q - U > 1 \,, \tag{33}$$

$$e_2 + d_3 + q - U > 1 \,, \tag{34}$$

$$g_3 + U > 1 \,, \tag{35}$$

Finally, for type VII, there are no alternatives, and so

THEOREM 7. *A set of type* VII, *will be a solution if and only if*

$$q - U > 1/3 \,. \tag{36}$$

5. Existence of solutions. We have, in § 4, given conditions for a set V, of the several types discussed, to be a solution. We now consider the more difficult problem of deciding the values of U and q for which such solutions exist. This will mean determining whether the conditions (4.7)-(4.14), (4.15)-(4.22), (4.28)-(4.31), (4.32)-(4.35), or (4.36) will be consistent with the remaining constraints of the problems.

We note first of all that, for the first six types of solution, a necessary condition is

$$2q - U > 1 \,. \tag{1}$$

In effect, this is due to the fact that, in any case,

$$d_1 + d_3 = e_2 + e_3 = f_1 + f_2 = q \,. \tag{2}$$

Now, looking at constraints (4.7)-(4.14) we see that (5.1) is implied, whatever g_1, g_2, and g_3 may be. Thus, if we have $g_1 = d_1$, $g_2 = f_2$, $g_3 = e_3$, we need only to add (4.7), (4.10) and (4.11), obtaining $6q - 3U > 3$. If, on the other hand, we should have $g_1 = d_1$, $g_2 = e_2$, $g_3 = d_3$, we would add (4.8), (4.9), (4.11) and (4.12) to obtain $8q - 4U > 4$. This disposes of types I, III, and VI, for all other cases reduce to one of these, by symmetry.

For type II, the same holds if we substitute the inequality $k < U/2$ in (4.15) and (4.16). For type IV, addition of (4.29) and (4.30) gives $4q - 2U > 2$. Finally, for type V, addition of (4.33) and (4.34) gives the same result.

Condition (5.1) is thus necessary for types I through VI. We look, however, for necessary and sufficient conditions.

Consider type I. We know that this can only exist if

$$2q \geqq 3U \,. \tag{3}$$

In addition to constraints (5.2), the points D, E, F must satisfy

$$d_1, d_3, e_2, e_3, f_1, f_2 \geqq U/2 \tag{4}$$

and it is clear that

$$d_1 = d_3 = e_2 = e_3 = f_1 = f_2 = q/2$$

will satisfy all the constraints (4.7)-(4.15), (5.2) and (5.4) whenever $q \geqq U$. But we must have $q \geqq U$ if (5.1) holds. Thus

THEOREM 8. *A game on the main diagonal will have a solution to type I, with $q < 1$, if and only if*

$$1/2 < U < 1 \,. \tag{5}$$

For such U, it will have such a solution for q satisfying

$$U + 1 < 2q \leqq 3U . \tag{6}$$

Proof. Condition (6) has been proved. Moreover, it is easy to see that (5) is necessary and sufficient for (6) to be feasible.

Consider next type II. We know that (3) must hold, as well as (2.10)-(2.11). Now, D, E, F must satisfy (2), and also

$$d_1, e_2, f_1, f_2 \geqq U - k \tag{7}$$

$$d_3, e_3 \geqq k \tag{8}$$

and we see that, if we choose $k = U/2 - \varepsilon$ (where ε is less than $2U - q - 1$) the vector

$$d_1 = d_3 = e_2 = e_3 = f_1 = f_2 = q/2$$

will satisfy constraints (4.15)-(4.22), (2), (7) and (8) whenever (1) and (3) hold. Thus

THEOREM 9. *A game on the main diagonal will have a solution of type* II, *with $q < 1$, if and only if*

$$1/2 < U < 1 . \tag{9}$$

For such U, it will have such a solution if q satisfies

$$U + 1 < 2q \leqq 3U . \tag{10}$$

Proof. Same as for Theorem 8.

We go on to type III. We know, first of all, that condition (3) is necessary. Moreover, we must have $3q - 4U \geqq 0$, as otherwise C will be outside the simplex of imputations, giving rise to a type V solution. Thus

$$2q/3 \leqq U \leqq 3q/4 . \tag{11}$$

Now D, E, F must satisfy (2), and also

$$d_1, e_2, f_1, f_2 \geqq 2U - q . \tag{12}$$

$$d_3, e_3 \geqq q - U . \tag{13}$$

We see that the vector

$$d_1 = e_2 = (3U - q)/2$$

$$d_3 = e_3 = (3q - 3U)/2$$

$$f_1 = f_2 = q/2$$

will satisfy all the constraints whenever (1) and (11) hold. Thus

THEOREM 10. *A game on the main diagonal will have a solution of type* III, *with* $q < 1$, *if and only if*

(14) $$1/2 < U < 3/4\ .$$

For such U *it will have such a solution for* q *satisfying*

(15) $$4U/3 \leqq q \leqq 3U/2$$

(16) $$2q > U + 1\ .$$

Consider next type IV. Once again, we know that we must have condition (3). However, k must satisfy conditions (2.17)–(2.18), which together imply the much stronger constraint:

(17) $$3q < 4U\ .$$

Finally, D and E must satisfy (2), and also

(18) $$d_1,\ e_2 \geqq U - k$$

(19) $$d_3,\ e_3 \geqq k\ .$$

Suppose now $U > 4q/5$. Then the vector

$$k = q - U + \varepsilon$$
$$d_1 = e_2 = U - k$$
$$d_3 = e_3 = q - U + k$$

will satisfy constraints (4.28)–(4.31), as well as (2), (18) and (19), whenever (1) holds.

Suppose, on the other hand, $U \leqq 4q/5$. Then, suppose $g_3 = d_3$ (analogous results will hold if we suppose $g_3 = e_3$). Adding constraints (4.28), (4.29) and twice (4.31), we find

$$6q - U - 2k > 4$$

which together with $k > q - U$ gives us

(20) $$q + U/4 > 1\ .$$

Now the vector

$$k = q - U + \varepsilon$$
$$d_1 = q - U/2$$
$$d_3 = U/2$$
$$e_2 = 3U/4$$
$$e_3 = q - 3U/4$$

is easily seen to satisfy the constraints, whenever (20) holds and $3q/4 \leqq U \leqq 4q/5$.

Considering the two possibilities, we see that, for $U > 4q/5$, we will have $2q - U > 1$ only if $U > 2/3$, whereas, for $U > 3q/4$, we can have $q + U/4 > 1$ whenever $U > 12/19$. Thus

THEOREM 11. *A game on the main diagonal will have a solution of type* IV, *with* $q < 1$, *if and only if*

$$\frac{12}{19} < U < 1 . \tag{21}$$

For $12/19 < U \leqq 2/3$, *it will have such solutions for* q *satisfying*

$$1 - \frac{U}{4} < q < 4U/3 , \tag{22}$$

whereas, for $2/3 < U \leqq 1$, *it will have such solutions for*

$$\frac{U+1}{2} < q < \frac{4U}{3} . \tag{23}$$

We go on to type V. As for type IV, (17) must hold. The points D and E must satisfy (2), and also

$$d_1, e_2 \geqq 2U - q \tag{24}$$

$$d_3, e_3 \geqq q - U . \tag{25}$$

It is clear that the vector

$$d_1 = e_2 = 2U - q$$

$$d_3 = e_3 = 2q - 2U$$

will satisfy all the constraints (4.7)–(4.14), (2), (24) and (25), whenever (1) and (17) hold. Thus

THEOREM 12. *A game on the main diagonal will have a solution of type V, with* $q < 1$, *if and only if*

$$3/5 < U < 1 . \tag{26}$$

It will have such solutions for q *satisfying*

$$4U/3 > q > \frac{U+1}{2} . \tag{27}$$

We go on to type VI. We know these solutions can exist only if $q/2 \leqq U \leqq 2q/3$. The constraints on D, E, F here are (2) and

(28) $$d_1, d_3, e_2, e_3, f_1, f_2 \geqq q - U\,.$$

It is not difficult to see that the vector

$$d_1' = d_3 = e_2 = e_3 = f_1 = f_2 = q/2$$

will satisfy constraints (4.7)–(4.14), (2) and (28) whenever (1) holds and $q/2 \leqq U \leqq 2q/3$. Thus we find

THEOREM 13. *A game on the main diagonal will have a solution of type* VI, *with* $q < 1$, for

(29) $$1/3 < U < 2/3\,.$$

It will have such a solution for q satisfying

(30) $$\tfrac{3}{2}U \leqq q \leqq 2U\,,$$

(31) $$q > \frac{1+U}{2}\,.$$

We go finally to type VII. Here, the situation is extremely simple, as there are no variables to worry about. We know we must have $q \geqq 2U$. From this and (4.36), we have

THEOREM 14. *A game on the main diagonal will have a solution of type* VII, *with* $q < 1$, *if and only if*

(32) $$0 \leqq U < 1/2\,.$$

For such U, solutions will exist for q satisfying

(33) $$q > U + 1/3$$

(34) $$q \geqq 2U\,.$$

6. Conclusion. This terminates, more or less, the study of discriminatory solutions. We find, however, that many assumptions have been made throughout. One is that $q < 1$, the other, that $U < q$. We clear this up by pointing out that, for $U \geqq q$, there can be solutions, if any, only of types I, IV, and V. If of type I, the solution would consist only of the three points:

$$A = (q/2,\, 0,\, q/2,\, 1 - q)$$

$$B = (0,\, q/2,\, q/2,\, 1 - q)$$

$$C = (q/2,\, q/2,\, 0,\, 1 - q)\,.$$

For $q < 1$, it is clear that (1/2, 1/2, 0, 0) is undominated by these. In effect, such domination could only be through $\{3,4\}$ by A or B. But

$$q/2 + 1 - q > 1 - q \geqq 1 - U = v(\{3,4\})$$

and so there is no domination.

As for a solution of types IV or V, this would consist only of the line AB, joining.

$$A = (q - k, 0, k, 1 - q)$$

and

$$B = (0, q - k, k, 1 - q)$$

and again $((1 - k)/2, (1 - k)/2, k, 0)$ is undominated as, for $i = 1, 2, 3$, we have $1 - q \geqq v(\{i, 4\})$.

We consider finally the case of $q = 1$. In this case, the problem has been solved (see, e.g., [4]). We will have solutions of the several types for:

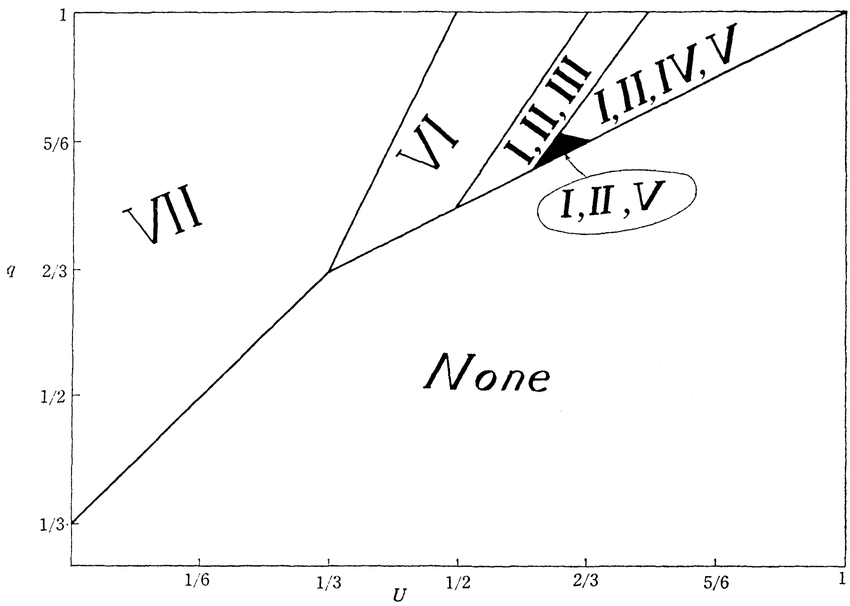

FIGURE 8. Types of Solutions Existing for Values of q, U.

(1)	$2/3 \leqq U \leqq 1$	(type I)
(2)	$2/3 \leqq U \leqq 1$	(type II)
(3)	$2/3 \leqq U \leqq 3/4$	(type III)
(4)	$3/4 \leqq U \leqq 1$	(type IV)
(5)	$3/4 \leqq U \leqq 1$	(type V)

$$(6) \qquad 1/2 \leqq U \leqq 2/3 \qquad \text{(type VI)}$$

$$(7) \qquad 0 \leqq U \leqq 1/2 \qquad \text{(type VII)}\,.$$

We conclude with Figure 8, which shows graphically the types of discriminatory solutions possible for all pairs (q, U) from 0 to 1.

BIBLIOGRAPHY

1. M. Hebert, *The doubly discriminatory solutions of the four-person constant-sum games*, Ann. of Math. Studies **40** (1964), 345-377.
2. G. Owen, *Discriminatory solutions of n-person games*, Proc. Amer. Math. Soc. **17** (1966), 653-657.
3. L. S. Shapley, *Notes on n-person games* VI: *On solutions that exclude one of more players*, RAND Corporation, RM-2533, 1960.
4. J. Von Neumann and O. Morgenstern, *The theory of games and economic behavior*, Princeton, 1944, 1947, 1953.

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