PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 2, 1970

AN ANALYSIS OF EQUALITY IN CERTAIN MATRIX INEQUALITIES, I

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In this paper we are concerned with analyzing the cases of equality in certain inequalities that relate the eigenvalues and main diagonal elements of hermitian matrices.

Let E_r denote the r^{th} elementary symmetric function of k variables ($E_0 = 1$). If $H=(h_{ij})$ is an n-square positive semi-definite hermitian matrix with eigenvalues $\gamma_1 \leqq \cdots \leqq \gamma_n$ and if $1 \leqq r \leqq k \leqq n$, then it is known that

$$E_r(h_{11}, \cdots, h_{kk}) \geqq E_r(\gamma_1, \cdots, \gamma_k) \; . \tag{1.1}$$

If $r > 1$ and at least r of $h_{11}, \cdots, h_{kk}$ are positive then (1.1) can be equality if and only if there exists a permutation $\varphi \in S_k$ such that

$$H = \operatorname{diag}(\gamma_{\varphi(1)}, \cdots, \gamma_{\varphi(k)}) \dotplus H_{n-k} \tag{1.2}$$

where H_{n-k} is $(n-k)$-square and $\dotplus$ denotes direct sum. Of course, if $r = k = n$ then (1.1) is the Hadamard determinant theorem:

$$\prod_{i=1}^{n} h_{ii} \geqq \det(H) \; . \tag{1.3}$$

If some $h_{ii} = 0$, then H is singular and (1.3) is equality. If $h_{ii} > 0, i=1, \cdots, n$, then the condition (1.2) yields the well-known criterion for equality in (1.3), namely $H = \operatorname{diag}(h_{11}, \cdots, h_{nn})$.

2. **Results.** Let $f(x) = f(x_1, \cdots, x_k)$ be a function defined for all nonnegative vectors $x \geqq 0$ (i.e., $x_i \geqq 0$, $i = 1, \cdots, k$). We shall assume that f is symmetric: $f(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) = f(x)$ for all $\sigma \in S_k$, the symmetric group of degree k. Let C_r denote the cone consisting of all $x \geqq 0$ with at least r positive components. The function f is said to be *strictly C_r-concave* if f is concave for $x \in C_r$ and if for x and y in C_r and $0 < \theta < 1$ the equality

$$f(\theta x + (1 - \theta)y) = \theta f(x) + (1 - \theta)f(y) \tag{2.1}$$

holds then it follows that $x \sim y$, i.e., x is a positive multiple of y. The usual definition of *strict concavity* requires that f be concave and that (2.1) holds if and only if $x = y$. We say that f is *C_r-positive* if: $f(x) > 0$ if and only if $x \in C_r$. Also, f is strictly *C_r-monotone* if $f(x + u) > f(x)$, $x \in C_r$, $u \geqq 0$, $u \neq 0$.

THEOREM 1. *Let $H = (h_{ij})$ be an n-square positive semi-definite* *hermitian matrix with eigenvalues* $0 \leqq \gamma_1 \leqq \cdots \leqq \gamma_n$. *Let* $1 \leqq r \leqq k \leqq n$. *Assume that* f *is symmetric, concave and nondecreasing in each variable. Let* $h_{\omega_t\omega_t}$, $t = 1, \cdots, k$, *be* k *main diagonal entries of* H. *Then*

$$f(h_{\omega_1\omega_1}, \cdots, h_{\omega_k\omega_k}) \geqq f(\gamma_1, \cdots, \gamma_k) . \tag{2.2}$$

Assume in addition that f *is strictly* C_r*-monotone, strictly* C_r*-concave and* C_r*-positive. If at least* r *of the* $h_{\omega_t\omega_t}$, $t = 1, \cdots, k$, *are positive then equality holds in* (2.2) *if and only if for some* $\varphi \in S_k$

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \qquad t = 1, \cdots, k , \tag{2.3}$$

and, in fact, in row and column ω_t, H *is* 0 *off the main diagonal,* $t = 1, \cdots, k$.

The inequality (2.2) is found in [3].

Proof. To begin with we can assume that $\omega_t = t$, $t = 1, \cdots, k$, and $h_{11} \leqq \cdots \leqq h_{kk}$. For, we can rearrange the main diagonal entries with a permutation similarity without affecting the eigenvalues. A trivial induction shows that for f strictly C_r-concave, $a^t \in C_r$, and $\theta_t > 0$, $t = 1, \cdots, m$, $\sum_{t=1}^{m} \theta_t = 1$, then

$$f\left(\sum_{t=1}^{m} \theta_t a^t\right) \geqq \sum_{t=1}^{m} \theta_t f(a^t) \tag{2.4}$$

and equality implies that $a^s \sim a^t$, $s, t = 1, \cdots, m$. Now there exists a unitary U such that $U^* \operatorname{diag}(\gamma_1, \cdots, \gamma_n) U = H$ and hence

$$h_{ii} = \sum_{j=1}^{n} |u_{ji}|^2 \gamma_j , \qquad i = 1, \cdots, n . \tag{2.5}$$

Since the matrix U is unitary we know that the matrix S whose (i, j) entry is $|u_{ji}|^2$, is doubly stochastic (d.s.). Thus (2.5) becomes

$$(h_{11}, \cdots, h_{nn}) = S(\gamma_1, \cdots, \gamma_n) . \tag{2.6}$$

Let $d = (h_{11}, \cdots, h_{nn})$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, and for any n-tuple x let $x[k]$ denote the truncated vector $(x_1, \cdots, x_k)$. If $\sigma \in S_n$ then $x^\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. By Birkhoff's theorem [1] let

$$S = \sum_{\sigma \in G} c_\sigma P_\sigma \tag{2.7}$$

where G is a subset of S_n, $c_\sigma > 0$, $\sigma \in G$, P_σ is an n-square permutation matrix corresponding to σ and $\sum_{\sigma \in G} c_\sigma = 1$. From (2.6), (2.7) and (2.4) we have

$$\begin{aligned} f(d[k]) &= f\left(\sum_{\sigma \in G} c_\sigma \gamma^\sigma[k]\right) \\ &\geqq \sum_{\sigma \in G} c_\sigma f(\gamma^\sigma[k]) . \end{aligned} \tag{2.8}$$

Consider a summand in (2.8) and choose $\mu_\sigma \in S_k$ so that

$$\sigma(\mu_\sigma(1)) < \cdots < \sigma(\mu_\sigma(k))$$

and hence

$$\gamma_{\sigma(\mu_\sigma(1))} \leqq \cdots \leqq \gamma_{\sigma(\mu_\sigma(k))} \,. \tag{2.9}$$

The symmetry of f implies that

$$f(\gamma^\sigma[k]) = f(\gamma^{\sigma\mu_\sigma}[k]) \,.$$

Now since $\sigma\mu_\sigma(t) \geqq t$, $t = 1, \cdots, k$, we know that

$$\gamma_{\sigma\mu_\sigma}(t) \geqq \gamma_t \,, \tag{2.10}$$

$t = 1, \cdots, k$. Then since f is nondecreasing in each variable we have

$$f(\gamma^\sigma[k]) \geqq f(\gamma_1, \cdots, \gamma_k) \tag{2.11}$$

and hence (2.8) becomes

$$f(d[k] \geqq f(\gamma_1, \cdots, \gamma_k) \,, \tag{2.12}$$

the required inequality (2.2).

Suppose equality holds in (2.12). Since $d[k] \in C_r$ we know that $f(d[k]) > 0$ and hence $f(\gamma[k]) > 0$. Thus $\gamma[k] \in C_r$. We also know that $f(\gamma^{\sigma\mu_\sigma}[k]) = f(\gamma[k])$ and in view of (2.10) it follows that

$$\gamma^{\sigma\mu_\sigma}[k] = \gamma[k] \,. \tag{2.13}$$

Setting $\mu_\sigma^{-1} = \nu_\sigma \in S_k$ in (2.13) we have

$$\gamma^\sigma[k] = (\gamma[k])^{\nu_\sigma} \,. \tag{2.14}$$

We must also have equality in (2.8) which because of the strict C_r-concavity implies that $\gamma^\sigma[k] \sim \gamma^\theta[k]$, σ, θ in G. In other words,

$$\gamma^\sigma[k] = a_\sigma \gamma^\tau[k]$$

for some fixed $\tau \in G$, $a_\sigma > 0$ all $\sigma \in G$. In view of (2.14)

$$\gamma^\sigma[k] = a_\sigma(\gamma[k])^{\nu_\tau}$$

so that

$$\begin{aligned} d[k] &= \sum_{\sigma \in G} c_\sigma \gamma^\sigma[k] \\ &= \sum_{\sigma \in G} c_\sigma a_\sigma (\gamma[k])^{\nu_\tau} \\ &= c(\gamma[k])^{\nu_\tau},\ c > 0 \,. \end{aligned}$$

The equality in (2.12) implies that

$$\begin{aligned} f(d[k]) &= f(\gamma[k]) \\ &= f(\gamma[k])^{\nu_\tau}) \end{aligned}$$

and thus

$$f(c(\gamma[k])^{\nu_\tau}) = f((\gamma[k])^{\nu_\tau})$$

or

$$f(c\gamma[k]) = f(\gamma[k]) .$$

Now $\gamma[k] \in C_r$ and hence by (2.1) $c = 1$. Thus

$$d[k] = (\gamma[k])^{\nu_\tau} . \tag{2.15}$$

Since $h_{11} \leqq \cdots \leqq h_{kk}$, (2.15) implies that

$$\gamma_{\nu_\tau(1)} \leqq \cdots \gamma_{\nu_\tau(k)} .$$

But $\gamma_1 \leqq \cdots \leqq \gamma_k$ and $\nu_\tau \in S_k$ and hence $\gamma_{\nu_\tau(t)} = \gamma_t$, $t = 1, \cdots, k$. In other words,

$$h_{ii} = \gamma_i , \qquad i = 1, \cdots, k . \tag{2.16}$$

Now we assert that (2.16) implies that the first k rows and columns of H are 0 off the main diagonal. To see this we observe that if $e_1 = (\delta_{11}, \cdots, \delta_{n1})$ and $u_1, \cdots, u_n$ are orthonormal eigenvectors of H corresponding to $\gamma_1, \cdots, \gamma_n$ respectively, then using the standard inner product in the vector space of complex n-tuples,

$$\begin{aligned} h_{11} &= (He_1, e_1) \\ &= \sum_{j=1}^{n} |(e_1, u_j)|^2 \gamma_j . \end{aligned} \tag{2.17}$$

Since $\gamma_1 = h_{11}$ we conclude from (2.17) that $(e_1, u_j) = 0$, if $\gamma_j > \gamma_1$. Suppose $\gamma_1 = \cdots = \gamma_r < \gamma_{r+1} \leqq \cdots \leqq \gamma_n$. Then $(e_1, u_j) = 0$, $j = r+1, \cdots, n$, and hence $e_1 \in \langle u_1, \cdots, u_r \rangle$, the space spanned by $u_1, \cdots, u_r$. But then $He_1 = \gamma_1 e_1$ and we conclude that the first column (and row) of H is 0 off the main diagonal. Since $\gamma_2, \cdots, \gamma_n$ are the eigenvalues of the submatrix obtained from H by deleting row and column 1, an obvious induction completes the proof.

Make the following choice for f:

$$f(x_1, \cdots, x_k) = E_r^{1/r}(x_1^q, \cdots, x_k^q) \tag{2.18}$$

where $0 < q \leqq 1$. We assert that for $r > 1$ or $r = 1, q < 1, f$ is strictly C_r-concave. For $0 < \theta < 1$ consider

$$\begin{aligned} f(\theta x + (1-\theta)y) &= E_r^{1/r}((\theta x_1 + (1-\theta)y_1)^q, \cdots, (\theta x_k + (1-\theta)y_k)^q) \\ &\geqq E_r^{1/r}(\theta x_1^q + (1-\theta)y_1^q, \cdots, \theta x_k^q + (1-\theta)y_k^q) \\ &\geqq \theta E_r^{1/r}(x_1^q, \cdots, x_k^q) + (1-\theta)E_r^{1/r}(y_1^q, \cdots, y_k^q) \\ &= \theta f(x) + (1-\theta)f(y) . \end{aligned} \tag{2.19}$$

In (2.19) we have used the monotonicity and C_r-concavity of $E_r^{1/r}$ [4], $r > 1$, and the strict concavity of $t^q, t \geqq 0$, for $r = 1$. When $q < 1$ the first inequality in (2.19) is strict unless $x = y$. If $q = 1, r > 1$, then the second inequality is strict unless $x \sim y$. In either event if (2.19) is equality then $x \sim y$ so that f is indeed strictly C_r-concave. Also, f is obviously strictly C_r-monotone and C_r-positive. We have

COROLLARY 1. *Let H satisfy the hypotheses of Theorem 1 and let $0 < q \leqq 1$. Then*

$$E_r(h^q_{\omega_1\omega_1}, \cdots, h^q_{\omega_k\omega_k}) \geqq E_r(\gamma_1^q, \cdots, \gamma_k^q) . \tag{2.20}$$

If at least r of the $h_{\omega_t\omega_t}$ are positive, $t = 1, \cdots, k$, then equality holds in (2.20) if and only if for some $\varphi \in S_k$,

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \qquad t = 1, \cdots, k ,$$

and H is 0 off the main diagonal in row and column ω_t, $t = 1, \cdots, k$.

We remark that if fewer than r of the $h_{\omega_t\omega_t}$ are positive then the left side of (2.20) is 0 and hence fewer than r of $\gamma_1, \cdots, \gamma_k$ are positive. If $r = k = n$ then (2.20) becomes

$$\prod_{j=1}^{n} h_{jj} \geqq \det H , \tag{2.21}$$

the Hadamard determinant theorem. If H is nonsingular and equality holds in (2.21) then Corollary 1 implies (since $h_{jj} > 0, j = 1, \cdots, n$) that $H = \operatorname{diag}(h_{11}, \cdots, h_{nn})$. If H is singular and equality holds in (2.21) then some $h_{jj} = 0$ and H has a zero row and column.

As another example consider the function

$$f(x) = E_r(x_1, \cdots, x_k)/E_{r-1}(x_1, \cdots, x_k)$$

for $x \in C_r$. We assert that f is strictly C_r-monotone, C-positive, and strictly C_r-concave. The C_r-positivity is obvious and the strict C_r-concavity is a result in [4]. To verify the strict C_r-monotonicity we show that for $x \in C_r$

$$\frac{\partial f}{\partial x_j} > 0 , \qquad j = 1, \cdots, k . \tag{2.22}$$

This will suffice since we are only interested in showing that $f(x + u) > f(x)$, $x \in C_r$, $u \geqq 0$, $u \neq 0$.

First observe that

$$E_r(x) = x_j E_{r-1}(\hat{x}_j) + E_r(\hat{x}_j) \tag{2.23}$$

where $E_r(\hat{x}_j)$ indicates the r^{th} elementary symmetric function of

$x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k$. Thus the sign of $\partial f/\partial x_j$ is the same as the sign of

$$E_{r-1}(x)E_{r-1}(\hat{x}_j) - E_r(x)E_{r-2}(\hat{x}_j) . \tag{2.24}$$

From (2.23) we see that (2.24) is equal to

$$\begin{aligned}(x_jE_{r-2}(\hat{x}_j) + E_{r-1}(\hat{x}_j))E_{r-1}(\hat{x}_j) - (x_jE_{r-1}(\hat{x}_j) + E_r(\hat{x}_j))E_{r-2}(\hat{x}_j)\\ = E^2_{r-1}(\hat{x}_j) - E_r(\hat{x}_j)E_{r-2}(\hat{x}_j) .\end{aligned}$$

Now it is known [2] that

$$E^2_{r-1}(\hat{x}_j) > E_r(\hat{x}_j)E_{r-2}(\hat{x}_j)$$

since at least $r - 1$ of the components of $\hat{x}_j$ are positive. We can now state

COROLLARY 2. *Let H satisfy the hypotheses of Theorem 1 and assume that at least $r-1$ of $\gamma_1, \cdots, \gamma_k$ are positive. Then*

$$\frac{E_r(h_{\omega_1\omega_1}, \cdots, h_{\omega_k\omega_k})}{E_{r-1}(h_{\omega_1\omega_1}, \cdots, h_{\omega_k\omega_k})} \geqq \frac{E_r(\gamma_1, \cdots, \gamma_k)}{E_{r-1}(\gamma_1, \cdots, \gamma_k)} . \tag{2.25}$$

If at least r of $\gamma_1, \cdots, \gamma_k$ are positive then the inequality (2.25) *is equality if and only if for some $\varphi \in S_k$*

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \qquad t = 1, \cdots, k$$

and H is 0 off the main diagonal in row and column ω_t, $t = 1, \cdots, k$.

Proof. First observe that if p of $\gamma_1, \cdots, \gamma_k$ are positive then H has at least $n - k + p$ positive eigenvalues. Hence since H is positive semi-definite we know that at most $n - (n - k + p) = k - p$ of the main diagonal elements can be 0. We conclude that any set of k main diagonal elements must contain at least p positive elements. It follows that both sides of (2.25) are defined. Also, if $p = r$ we obtain the stated conditions for equality by applying Theorem 1.

We can derive an immediate consequence of Theorem 1 by replacing the matrix H by X^*HX where X is any n-square unitary matrix. The main diagonal entries of X^*HX are (Hx_j, x_j), $j = 1, \cdots, n$ where x_j is the jth column of X.

COROLLARY 3. *Let H and f be as in Theorem 1. Then for any set of k orthonormal vectors $x_1, \cdots, x_k$,*

$$f((Hx_1, x_1), \cdots, (Hx_k, x_k)) \geqq f(\gamma_1, \cdots, \gamma_k) . \tag{2.26}$$

If at least r of the inner products (Hx_j, x_j), $j = 1, \cdots, k$, are positive *then* (2.26) *is equality if and only if*

$$Hx_j = \gamma_{\varphi(j)}x_j\ , \qquad j = 1, \cdots, k\ , \tag{2.27}$$

for some $\varphi \in S_k$, i.e., $x_1, \cdots, x_k$ *are an orthonormal set of eigenvectors corresponding to* $\gamma_1, \cdots, \gamma_k$ *in some order.*

Proof. Let X be a unitary matrix whose first k columns are $x_1, \cdots, x_k$. The result (2.26) follows from Theorem 1 applied to X^*HX. If equality holds and if r of the inner products $(Hx_1, x_1), \cdots, (Hx_k, x_k)$ are positive then X^*HX is 0 off the main diagonal in row and column j, $j = 1, \cdots, k$, and $(X^*HX)_{jj} = \gamma_{\varphi(j)}$, $j = 1, \cdots, k$, for an appropriate $\varphi \in S_k$. This completes the proof.

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Received May 8, 1969, and in revised form December 15, 1969. The work of the first author was supported by the U.S. Air Force Office of Scientific Research under grant AFOSR 698-67.

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