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ON THE EXISTENCE QUESTION FOR A FAMILY OF PRODUCTS

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Let X be a topological space and let P and Q be finite dimensional linear subspaces of $C(X)$. Since the set $PQ = \{pq: p \in P, q \in Q\}$ is a subset of a finite dimensional linear subspace of $C(X)$, existence of best approximations from PQ is assured if and only if PQ is closed. If $p \in P, q \in Q$, and $pq=0$ imply that $p=0$ or $q=0$, then PQ is shown to be closed. An example shows that PQ is not closed in general.

In an important paper by B. Boehm, [1], the problem of existence of best approximations from certain nonlinear families is discussed. One of these families is formed by taking products of elements of two finite dimensional linear subspaces of $C(X)$, where X is a topological space. In a seminar Professor H. Loeb pointed out an error in Boehm's existence proof for this family. Professor Loeb then posed the question whether the result was correct and if it was not to develop reasonable hypotheses to insure existence.

In this paper we consider these questions and show that Boehm's hypotheses have to be modified.

Throughout this paper X will denote a topological space and $C(X)$ the vector space of all bounded, continuous, real-valued functions defined on X. The symbol $\|\cdots\|$ will represent the uniform norm defined on $C(X)$ by $\|f\| = \sup\{|f(x)|: x \in X\}$.

One question under consideration is that of existence of best approximations from the set

$$PQ = \{pq: p \in P \text{ and } q \in Q\}$$

where P and Q are finite dimensional linear subspaces of $C(X)$. (Here $(pq)(x) = p(x)\cdot q(x)$.) First of all it is easy to see that PQ is a subset of a finite dimensional linear subspace of $C(X)$. In fact, if $\{g_1, \cdots, g_m\}$ is a basis of P and $\{h_1, \cdots, h_n\}$ is a basis of Q then it is clear that PQ is contained in the linear space generated by the set of products of functions $\{g_i h_j: i = 1, \cdots, m \text{ and } j = 1, \cdots, n\}$. The dimension of this linear space is at most equal to mn. Now, it is known that best approximations exist from a subset of a finite dimensional linear subspace if and only if the subset is closed. But, as the following example shows, the set PQ is not closed in general.

EXAMPLE 1. Let $P = \{ag_1 + bg_2: a, b \text{ real}\}$ and $Q = \{ch_1 + dh_2: c, d \text{ real}\}$ be finite dimensional linear subspaces of $C[0, 5]$ with basis functions described by the graphs in Figure 1.

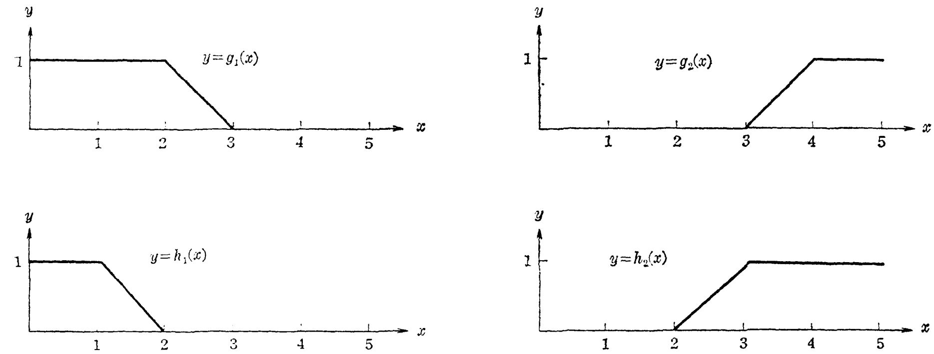

FIGURE 1.

Observe that

(1) $g_1h_1 = h_1$, $g_2h_1 = 0$, and $g_2h_2 = g_2$ and that
(2) h_1, g_1h_2 and g_2 are linearly independent.

Now, consider the sequence $\{p_nq_n\}_{n=1}^{\infty}$ in PQ in which

$$p_n = g_1 + ng_2$$

and

$$q_n = h_1 + \frac{1}{n}h_2$$

for each n. By (1) it follows that

$$\begin{aligned} p_nq_n &= (g_1 + ng_2)\left(h_1 + \frac{1}{n}h_2\right) \\ &= g_1h_1 + \frac{1}{n}g_1h_2 + ng_2h_1 + g_2h_2 \qquad (*) \\ &= h_1 + \frac{1}{n}g_1h_2 + g_2 \end{aligned}$$

for each n. Thus

$$p_nq_n \longrightarrow h_1 + g_2$$

as $n \rightarrow \infty$ so $h_1 + g_2$ is in the closure of PQ. If $h_1 + g_2$ is contained in PQ then there exist real coefficients a, b, c and d such that

$$(ag_1 + bg_2)(ch_1 + dh_2) = h_1 + g_2 .$$

By (1) the above equation reduces to

$$ach_1 + adg_1h_2 + bdg_2 = h_1 + g_2$$

and by (2) to the system

$$ac = 1$$
$$ad = 0$$
$$bd = 1$$

which has no solution. Thus $h_1 + g_2$ is not in PQ and PQ is not closed.

The above result shown that Boehm's Theorem 4, [1], is incorrect. The fact that $g_2h_1 = 0$ even though $g_2 \neq 0$ and $h_1 \neq 0$ plays an important role in Example 1, as is apparent from $(*)$. As it turns out, one modification to Boehm's hypotheses which implies that PQ is closed is to rule out the possibility of a product in PQ being zero when neither of the factors is zero.

THEOREM 1. *Let P and Q be finite dimensional linear subspaces of $C(X)$. If $p \in P$, $q \in Q$, and $pq = 0$ imply that $p = 0$ or $q = 0$ then PQ is closed.*

Proof. Let g be a function in the closure of PQ, and let $\{p_nq_n\}_{n=1}^{\infty}$ be a sequence in PQ such that

$$||g - p_nq_n|| \longrightarrow 0 \tag{3}$$

as $n \rightarrow \infty$. If $g = 0$ then certainly $g \in PQ$ since $0 \in P$ and $0 \in Q$. So, assume that $g \neq 0$. Then with no loss of generality it may be assumed that $p_nq_n \neq 0$ and that $||p_n|| = 1$ for each n. Since closed and bounded subsets of finite dimensional linear spaces are compact, it suffices to show that a subsequence of the sequence $\{q_n\}_{n=1}^{\infty}$ is bounded. Assume the contrary. Then by going to subsequences, if necessary, and letting $n \rightarrow \infty$ it follows that

$$\frac{p_nq_n}{||q_n||} \longrightarrow 0\,. \tag{4}$$

By the preceding remark on compactness there exist nonzero functions $p \in P$ and $q \in Q$ such that

$$\frac{p_nq_n}{||q_n||} \longrightarrow pq \tag{5}$$

by going to further subsequences, if necessary, and letting $n \rightarrow \infty$. But (4) implies that $pq = 0$ which is a contradiction.

Clearly, in Theorem 1, $C(X)$ can be replaced by any real or complex normed algebra.

A counter example to Boehm's Theorem 5, [1], exists using ordinary polynominals and corresponding rational functions.

In a forthcoming paper we will discuss further the existence problem and also the characterization question for this setting.

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