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ON MATRICES WITH A RESTRICTED NUMBER OF DIAGONAL VALUES

J. E. H. ELLIOTT

This note confirms the following conjecture of Marcus: Let $A = (a_{ij})$ be an $n \times n$ matrix of strictly positive entries with at most $(n-1)$ distinct diagonal values, then A is singular. We also show that there exist matrices with strictly positive entries with n diagonal values which are nonsingular.

DEFINITIONS. If A is an $n \times n$ matrix and σ is a permutation of $\{1, 2, \cdots, n\}$, then the product $a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$ is called the *diagonal* of A corresponding to σ.

If A_1, A_2 are two $n \times n$ matrices, then A_1 is called a *diagonate* of A_2 if A_1 can be obtained from A_2 by a finite number of operations of the following kinds:

(i) Multiplication of all entries of some row, (or column) by some $c > 0$.

(ii) Interchange of any two rows (or columns).

The notation $A[\mu \mid \gamma]$, $A(\mu \mid \gamma)$ is that of [1].

PRELIMINARY REMARKS. (i) The property of being a diagonate is an equivalence relation.

(ii) If a matrix is singular (nonsingular), then each of its diagonates is singular (nonsingular).

(iii) If a matrix A_1 has diagonal values $\rho_1 < \rho_2 < \cdots < \rho_r$ then a diagonate A_2 of A_1 has diagonal values $k\rho_1 < k\rho_2 < \cdots k\rho_r$, where $k = k(A_2)$, and $|\det A_1| = |k \det A_2|$.

(iv) If a matrix has strictly positive (positive) entries, then each of its diagonates has strictly positive (positive) entries.

LEMMA. *If $X = (x^{e(i,j)})$ is an $n \times n$ matrix with entries in an extension $F(x)$ of the real field F, where $e(i, j)$ are nonnegative rational integers $i, j = 1, 2, \cdots n$ and $e(1, j) = 0$ for $j = 1, 2, \cdots n$, then*

$\det X = (x-1)^{n-1} g(x)$, *where $g(x)$ is a polynomial in x with rational integral coefficients.*

The proof of the lemma is by induction. The result is trivial for $n = 2$. The result is therefore assumed to hold for all $n < N$, and $N > 2$. If $n = N$, subtracting the first row of X from the second and expanding X by its second row, we have

$$\det X = \sum_{j=1}^{n} (-1)^j \{x^{e(2,j)} - 1\} \det X(2 \mid j) ;$$

but each of the matrices $X(2 \mid j)$ is of the form of the matrix of the hypothesis, and therefore by the induction assumption we have

$$\det X(2 \mid j) = (x - 1)^{n-2} g_j(x) ,$$

where $g_j(x)$ is a polynomial in x with rational integral coefficients. Thus

$$\det X = \sum_{j=1}^{n} (-1)^j \{x^{e(2,j)} - 1\}(x - 1)^{n-2} g_j(x) = (x - 1)^{n-1} g(x) .$$

We are now in a position to prove the conjecture.

The conjecture is proved below by induction on the order of the matrix. Therefore we first prove the theorem for a 3×3 matrix.

THEOREM 1. *If A_α is a 3×3 matrix of strictly positive entries with at most two distinct diagonal values, then A_α is singular.*

To prove this, it is supposed that A_α is nonsingular: then there exist nonsingular minors $A_\alpha(i \mid j)$ with diagonal values $\rho_1(i, j) < \rho_2(i, j)$. Consequently there exists a diagonate A_β of A_α where the ratio $\lambda = \rho_2(1, 1)/\rho_1(1, 1)$ is maximal, and A_β has two distinct diagonal values $\gamma_{11}\rho_1(1, 1)$, $\lambda\gamma_{11}\rho_1(1, 1)$. Thus there exists a diagonate A_γ or A_β such that $\gamma_{3i} = \gamma_{i3} = 1$ for $i = 1, 2, 3$, $\gamma_{22} = \lambda$ where $A_\gamma = (\gamma_{ij})$. Since A_α is nonsingular A_γ is also nonsingular, and λ retains its maximality property in A_γ. Now if d is the entry $A_\gamma(i, 3 \mid j, 3)$ where $i \neq 3$, $j \neq 3$, then $\gamma_{ij}d$ and d are both diagonal values, so consideration of their ratio shows that $\gamma_{ij} = \lambda, 1$ or λ^{-1}. Consideration of the minors $A_\gamma[1, 3 \mid 2, 3]$ and $A_\gamma[1, 2 \mid 2, 3]$ shows, by the maximality property of λ, that γ_{21}, γ_{12} are no less than 1. Putting $\gamma_{11} = 1$ therefore, since no columns (rows) are equal, yields $\gamma_{21} = \gamma_{12} = \lambda$. This gives a contradiction, as the matrix now has three distinct diagonal values $1, \lambda$ and λ^2. If $\lambda_{11} = \lambda^{-1}$, then A_γ has distinct diagonal values λ, λ^{-1}, and a consideration of their ratio leads to a contradiction. We must therefore have $\gamma_{11} = \lambda$, and so A_γ has diagonal values λ, λ^2. However, since γ_{21} and γ_{12} are also diagonal values each equal to 1, or λ, then $\gamma_{12} = \gamma_{21} = \lambda$, and again since A_γ is nonsingular we have a contradiction. But this has exhausted all possibilities for the value of γ_{11} and so the proof of Theorem 1 is complete.

We are now in a position to prove the conjecture for all n.

THEOREM 2. *If A_α is an $n \times n$ matrix of strictly positive entries with at most $(n-1)$ distinct diagonal values then A_α is singular.*

The proof of this theorem is by induction on n. The result is trivial for $n = 2$, and it has been proved for $n = 3$. Therefore we assume the theorem to hold for all $n < N$, where $N > 3$. It is supposed that A_α is an $N \times N$ matrix of the diagonate class $A = \{A_\omega; \omega \in \Omega\}$. The proof is by contradiction; we assume that A_α is nonsingular. By the Expansion Theorem of Laplace, [1], given two rows r, s of A_α there exist two columns t, u such that $A_\alpha[r, s \mid t, u)$ and $A_\alpha(r, s \mid t, u)$ are both nonsingular. It then follows from the induction assumption that the matrix $A_\alpha[r, s \mid t, u]$ has at least two distinct diagonal values $\mu_1 < \mu_2$, and the matrix $A_\alpha(r, s \mid t, u)$ has at least $(N-2)$ distinct diagonal values $\rho_1 < \rho_2 < \cdots < \rho_{N-2}$. Therefore A_α must have at least the $(N-1)$ distinct diagonal values $\mu_1\rho_1 < \mu_2\rho_1 < \mu_2\rho_2 < \cdots < \mu_2\rho_{N-2}$. However A_α has at most $N-1$ distinct diagonal values, and so these diagonal values must also be exactly the values

$$\mu_1\rho_1 < \mu_1\rho_2 < \cdots < \mu_1\rho_{N-2} < \mu_2\rho_{N-2} .$$

It therefore follows that

$$\frac{\mu_2}{\mu_1} = \frac{\rho_2}{\rho_1} = \cdots = \frac{\rho_{N-2}}{\rho_{N-3}} > 1 .$$

Hence if λ denotes the ratio μ_2/μ_1, then the matrix A_α has for its $(N-1)$ distinct diagonal values exactly the $(N-1)$ diagonal values $c < \lambda c < \cdots < \lambda^{N-2}c$, where $c = \mu_1\rho_1$. Now there exists $A_\beta = (a_{ij}) \in A$ such that $a_{i1} = a_{1i} = 1$ for $i = 1, 2, \cdots, N$, and A_β has diagonal values $k < \lambda k < \cdots < \lambda^{N-2}k$ for some $k > 0$. If d is any diagonal value of $A_\beta(1, i \mid 1, j)$ then $a_{ij}d$, and d are diagonal values of A_β and thus a_{ij} is an integral power of λ. A division of the j-th row of A_β by $\min\{a_{ij}; i = 1, 2, \cdots N\}$ for $j = 2, 3, \cdots N$, yields a matrix $A_\gamma \in A$, $A_\gamma = (\gamma_{ij})$ such that $\gamma_{ij} = \lambda^{e(i,j)}$ where $e(i, j)$ is a nonnegative rational integer for $i, j = 1, 2, \cdots N$, $e(1, j) = 0$ for $j = 1, \cdots N$, and A_γ has diagonal values

$$\lambda^h < \lambda^{h+1} < \cdots < \lambda^{h+N-2} .$$

Now let E denote the $N \times N$ matrix with (i, j)th entry $x^{e(i,j)}$, where x is transcendental over the real field. By the lemma, $\det E = (x-1)^{N-1}g(x)$, where $g(x)$ is a polynomial with rational integral coefficients. However E has exactly the diagonal values

$$x^h < x^{h+1} < \cdots < x^{h+N-2}$$

and thus $\det E = x^h\{b_0 + b_1x + \cdots b_{N-2}x^{N-2}\} = (x-1)^{N-1}g(x)$ where b_i,

$i = 0, 1, \cdots N-2$ is a rational integer. This however implies that $b_0 = b_1 = \cdots = b_{N-2} = 0$, and thus

$$\det A_\gamma = \lambda^h\{b_0 + b_1\lambda + \cdots + b_{N-2}\lambda^{N-2}\} = 0 .$$

We therefore have A_α, A_γ two matrices of the same diagonate class one nonsingular and one singular. This is the required contradiction which completes the proof of the conjecture. We can also conclude the result below.

COROLLARY. *If an $n \times n$ matrix A with strictly positive entries has at most r distinct diagonal values and $r < n$, then* $\operatorname{rank}(A) \leqq r$.

To show that an $n \times n$ matrix of strictly positive entries need not be singular if it takes on as few as n diagonal values, we may consider the $n \times n$ matrix $C = (c_{ij})$, where $c_{ii} = k$ for $i = 2, 3, \cdots n$, and $c_{ij} = \lambda$ otherwise; and where k, λ are positive integers such that $k > \lambda$. Then $\det C = \lambda(k-\lambda)^{n-1} \neq 0$.

REFERENCES

1. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon Inc., Boston, 1964.

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UNIVERSITY COLLEGE, CORK