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ON PARTIAL HOMOMORPHISMS OF SEMIGROUPS

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Let S be a semigroup and T be a semigroup with zero ($T=T^0$). An ideal extension of S by T is a semigroup V containing S as an ideal and such that the Rees quotient V/S is isomorphic to T. A mapping α from $T^*=T-\{0\}$ into S is said to be a partial homomorphism, if $t_1, t_2 \in T^*$, $t_1t_2 \neq 0$ implies $(t_1t_2)\alpha=(t_1\alpha)(t_2\alpha)$. Every partial homomorphism from T^* into S gives rise to an ideal extension of S by T. Further, in certain cases every ideal extension of S by T is obtained in this way. In this paper a characterization is given for all partial homomorphisms from T^* into S.

It is not known in general when all extensions of S by T are determined by the partial homomorphisms of T^* into S. Clifford has shown this to be the case when S has an identity (see [2, §4.4]). Further results in this direction have been obtained by Warne [4] and Petrich [3]. The partial homomorphisms of a completely 0-simple semigroup into an arbitrary semigroup have been determined by Clifford [1].

An element x of a semigroup S is said to be *prime* if x does not belong to S^2. S is said to have *unique factorization* if every nonzero element of S can be written uniquely as a product of powers of primes. Of course, if S is not commutative, we must take the order of the factors into account. We define the *kernel* of a homomorphism into a semigroup with zero to be the complete inverse image of zero.

THEOREM 1. *A [commutative] semigroup S has unique factorization if and only if S is free [commutative] or the Rees quotient of a free [commutative] semigroup.*

Proof. Suppose S has unique factorization, and let X be the set of primes of S. If $0 \notin S$, then clearly S is free [commutative] on X. So assume $0 \in S$, and let F_X be the free [commutative] semigroup on X with homomorphism ϕ from F_X onto S such that $x\phi = x$ for all $x \in X$ [2, p. 41]. Let K be the kernel of ϕ. Since S has unique factorization, ϕ must be one-to-one on $F_X - K$, so S is isomorphic to the Rees quotient F_X/K. The converse is obvious.

COROLLARY 2. *If $S = S^0$, then there exists a semigroup U with unique factorization and a homomorphism from U onto S with trivial kernel.*

Proof. There exists a free semigroup F which is homomorphic onto S with kernel K. Set $U = F/K$ and use Theorem 1.6 of [2].

THEOREM 3. *Let U be a semigroup with unique factorization and let X be the set of primes of U. Let S be any semigroup. Then any mapping α from X into S can be extended to a partial homomorphism of U^* into S.*

Proof. Omitted.

We denote by π_α the equivalence relation induced by a mapping α on its domain and use $\leqq$ for the usual partial ordering of relations on a set.

THEOREM 4. *Let $T = T^0$ and S be semigroups. By Corollary 2 there exists a semigroup U with unique factorization and a homomorphism ϕ from U onto T with trivial kernel. Let α be any partial homomorphism from U^* into S such that $\pi_\phi \leqq \pi_\alpha$ on U^* and define $\alpha': T^* \rightarrow S$ as follows. If $y \in T^*$ then there exists an $x \in U^*$ such that $y = x\phi$ and we define $y\alpha' = x\alpha$. Then α' is a partial homomorphism from T^* into S. Conversely every partial homomorphism of T^* into S is determined in this manner. Finally, the mapping $\alpha \rightarrow \alpha'$ is one-to-one.*

Proof. α' is well defined since $\pi_\phi \leqq \pi_\alpha$ on U^*, and it is a partial homomorphism since α is. Conversely, if α' is a partial homomorphism from T^* into S, then define $x\alpha = x\phi\alpha'$ for $x \in U^*$. If $x_1, x_2 \in U^*$ with $x_1x_2 \neq 0$, then $(x_1x_2)\phi\alpha' = ((x_1\phi)(x_2\phi))\alpha'$ and this in turn is equal to $(x_1\phi\alpha')(x_2\phi\alpha')$ since ϕ has trivial kernel. Thus α is a partial homomorphism from U^* into S such that $\pi_\phi \leqq \pi_\alpha$ on U^*.

Now let α, β be partial homomorphisms from U^* into S such that $\pi_\phi \leqq \pi_\alpha, \pi_\phi \leqq \pi_\beta$ on U^* and $\alpha' = \beta'$. Thus for all $x \in U^*$, $x\phi\alpha' = x\phi\beta' \Rightarrow x\alpha = x\beta$ so $\alpha = \beta$ and the mapping is one-to-one.

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