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ON THE NUMBER OF FINITELY GENERATED 0-GROUPS

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Let K be a class of relational systems of a fixed similarity type, $\mathfrak{n}$ an infinite cardinal. A system $\mathfrak{A}$ of cardinality $\mathfrak{n}$ is $(\mathfrak{n}, K)$-weakly universal if each system in K of cardinality at most $\mathfrak{n}$ is isomorphically embeddable in A. The object of this note is to construct $2^{\aleph_0}$ nonisomorphic finitely generated 0-groups and hence answer in the negative the following problem attributed to B. H. Neumann. Is there a group which is $(\aleph_0, K_1)$-weakly universal, where K_1 is the class of 0-groups?

If $\mathfrak{A}$ is $(\mathfrak{n}, K)$-weakly universal and also a member of K, then $\mathfrak{A}$ is $(\mathfrak{n}, K)$-universal. It is known that $(\mathfrak{n}, K)$-universal systems exist for many classes K and cardinals $\mathfrak{n}$. In particular, Morley and Vaught established a useful condition for the existence of $(\mathfrak{n}, K)$-universal systems for K an elementary class, $\mathfrak{n}$ an appropriate cardinal (see [7]). However there are no theorems of wide applicability concerning the existence of $(\aleph_0, K)$-universal systems; here the structure of the systems in K must be carefully analyzed. To illustrate this, consider the classes K_1 of 0-groups; K_2 of abelian 0-groups (i.e., torsion free abelian groups); K_3 of ordered groups (i.e., groups of type $\langle H, \cdot, \leqq \rangle$ where $\langle H, \cdot \rangle$ is an 0-group linearly ordered by $\leqq$); K_4 of abelian ordered groups. By applying the results in [7], (assuming the generalized continuum hypothesis), it is easily seen that there exists an $(\mathfrak{n}, K_i)$-universal system for all $\mathfrak{n} > \aleph_0$ and $i = 1, 2, 3,$ or 4.

The situation for $\mathfrak{n} = \aleph_0$ is more complicated. There is an $(\aleph_0, K_2)$-universal group (see [1, p. 64]). However, there is no ordered group which is $(\aleph_0, K_4)$-weakly universal and hence there is no $(\aleph_0, K_3)$-universal group. This follows readily from the fact that the free abelian group on two generators has $2^{\aleph_0}$ nonisomorphic orders (see [2, p. 50]). Theorem 2, which establishes the nonexistence of a group which is $(\aleph_0, K_1)$-weakly universal, solves a problem of B. H. Neumann (see [2, p. 211, Problem 17]).

1. Definitions. An 0-group is a group G for which there exists a linear ordering relation $\leqq$ on G satisfying the following condition: $a \leqq b$ implies $c\,a\,d \leqq c\,b\,d$ for all $a, b, c, d \in G$. For a group G the commutator of x and y in G is denoted $[x, y] = x^{-1}\,y^{-1}\,xy$; for subsets A and B of G, $[A, B]$ is the subgroup of G generated by $\{[a, b] : a \in A, b \in B\}$; $G' = [G, G]$; $G'' = [G', G']$. Let F be the free

group generated by a set X; a set R of equations of the form $w_1 = w_2$, where w_1 and w_2 are words in F, is a set of relations in X. A group G generated by the set X is given by a set R of defining relations if the following conditions are satisfied.

(i) R is a set of relations in X.

(ii) Let φ_G be the unique homomorphism from F onto G which extends the identity map on X. Then the kernel of φ_G is the normal subgroup of F generated by $\{w_1w_2^{-1}\colon w_1 = w_2 \in R\}$.

2. Finitely generated 0-groups.

In [4], P. Hall constructs $2^{\aleph_0}$ nonisomorphic finitely generated groups H each having torsion-free center and satisfying the condition $[H'', H] = 1$. We will show that these groups are also 0-groups.

LEMMA 1. (*B. H. Neumann*). *Let G be an 0-group generated by a set X and given by a set R of defining relations; let H be a group generated by the set $\{a\} \cup X$ where $a \notin X$, with the relations R and $[a^{-n}\, b\, a^n, b'] = 1$ for all $b, b' \in X$, $n = 1, 2, 3, \cdots$ as a set of defining relations. Then H is an 0-group.*

Proof. See [6, pp. 10-11].

The next lemma is a slight variant of von Dyck's Theorem (see [5, p. 130]).

LEMMA 2. *Let G be a group generated by a set X, given by a set R of defining relations; let H be a group generated by X, given by the set $R \cup S$ of defining relations. Then H is isomorpic to G/N* where *N is the normal subgroup of G generated by*

$$\{\varphi_G(w_1w_2^{-1})\colon w_1 = w_2 \in S\} .$$

THEOREM 1. *There exist $2^{\aleph_0}$ nonisomorphic finitely generated 0-groups.*

Proof. In his construction, P. Hall used a group G satisfying the following conditions:

(1) G is generated by the set $\{a, b\}$. For notational convenience we will write $b = b_0$ and

$$b_i = a^{-i}\, ba^i \qquad i = 0, \pm 1, \pm 2, \cdots .$$

G is given by the defining relations

$$[[b_i, b_j], b_k] = 1 \qquad i, j, k = 0, \pm 1, \pm 2, \cdots$$

$$[b_j, b_i] = [b_{j+k}, b_{i+k}] \qquad i, j, k = 0, \pm 1, \pm 2, \cdots$$

and $i < j$.

(2) the center Z of G is free abelian with generators

$$\{[b_i, b] : i = 1, 2, 3, \cdots\} .$$

Let C be a denumerable torsion-free abelian group. Appealing to [4] (p. 433), we find that there is a set $\{H_\iota : \iota < 2^{\aleph_0}\}$ of nonisomorphic groups satisfying the following conditions:

(3) the center C_ι of H_ι is isomorphic to C and H_ι/C_ι is isomorphic to G/Z;

(4) $[H_\iota'', H_\iota] = 1$ and each H_ι is generated by two elements.

As is known (see [3, p. 94]), a group H is an 0-group if both its center C and the factor group H/C are 0-groups. But by (3), each C_ι is an 0-group and H_ι/C_ι is isomorphic to G/Z. Hence, to verify that each H_ι is an 0-group it suffices to show that G/Z is an 0-group.

Let B be a group generated by the set $\{a, b\}$ and given by the defining relations occurring in (1) and the relations

$$[b_k, b_0] = 1 \qquad \text{for } k = 1, 2, 3, \cdots . \tag{5}$$

By Lemma 2, B is isomorphic to G/N, where N is the normal subgroup of G generated by

$$\{\varphi_G[b_k, b_0] : k = 1, 2, 3, \cdots\} = \{[b_k, b_0] \in G : k = 1, 2, 3, \cdots\} .$$

Applying (2), we have $N = Z$ and B is isomorphic to G/Z. Furthermore, B is given by the defining relations (5) alone. For if we assume $j > 0$ and use (5), then we have:

$$\begin{aligned}(a^j b a^{-j})b &= (a^j b a^{-j}) \; b(a^j a^{-j}) \\ &= a^j(b(a^{-j} b a^j) a^{-j}) \\ &= a^j((a^{-j} b a^j) b a^{-j}) \\ &= b(a^j b a^{-j}) .\end{aligned}$$

Thus,

$$[b_k, b_0] = 1 \qquad \text{for } k = 0, \pm 1, \pm 2, \cdots .$$

A similar computation yields

$$[b_i, b_j] = 1 \qquad \text{for } i, j = 0, \pm 1, \pm 2 \cdots .$$

Hence the relations in (1) hold trivially.

Since the free group with generating set $\{b\}$ is an 0-group, we can infer from Lemma 1 that the group B which is generated by

$\{a, b\}$ and given by the defining relations (5) is an 0-group; i.e. G/Z is an 0-group.

Since a countable group has only countably many finitely generated subgroups, we obtain our conclusion:

THEOREM 2. *There does not exist a group which is* $(\aleph_0, K_1)$-*weakly universal.*

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