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NONTANGENTIAL HOMOTOPY EQUIVALENCES

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The purpose of this paper is to apply surgery techniques in a simple, geometric way to construct manifolds which are nontangentially homotopy equivalent to certain π-manifolds. Applying this construction to an H-manifold of the appropriate type yields an infinite collection of mutually nonhomeomorphic H-manifolds, all nontangentially homotopy equivalent to the given one.

The theorem proved is the following: If N^{4k} is a smooth, closed, orientable π-manifold and L^m is a smooth, closed, simply connected π-manifold, there is a countable collection of smooth, closed manifolds $\{M_i\}$ satisfying (1) no M_i is a π-manifold, (2) each M_i is homotopy equivalent but not homeomorphic to $N \times L$, (3) M_i is not homeomorphic to M_j if $i \neq j$.

1. Construction of the surgery problem. In [2] Milnor describes a $(2k-1)$-connected, bounded π-manifold of dimension $4k$ and Hirzebruch index 8 ($k \geqq 2$). This manifold, which we denote by Y^{4k}, is obtained by plumbing together 8 copies of the tangent disk bundle of S^{2k} according to a certain scheme. This implies that Y has the homotopy type of a bouquent of eight $2k$-spheres. The only other property of Y which we shall need is that ∂Y is a homotopy sphere. Let r be the order of ∂Y^{4k} in the group of homotopy spheres bP_{4k} [3] and take W^{4k} to be the r-fold connected sum along the boundary of Y^{4k}. By the choice of r, ∂W is diffeomorphic to S^{4k-1}. Attaching a $4k$-disk to W by a diffeomorphism along the boundary, we obtain a closed, smooth manifold $\hat{W}$, which is $(2k-1)$-connected and has index $8r$. By the Hirzebruch index theorem $\hat{W}$ is not a π-manifold, but is almost parallelizable.

Define $f: W^{4k} \to D^{4k}$ by the identity on the boundary, stretching a collar of ∂W over D^{4k}, and sending the remainder of W to a point. This gives a degree 1 map $f: (W, \partial W) \to (D^{4k}, \partial D^{4k})$ which is tangential since both W and D^{4k} are π-manifolds. f is already a homotopy equivalence on the boundary, so we have a surgery problem in the bounded case. The connectedness of W implies that f is already an isomorphism in homology below the middle dimension. However the kernel of f_* in dimension $2k$ is $\underbrace{Z \oplus \cdots \oplus Z}_{8r}$ and the index of the kernel is the index of W which is $8r$. Thus it is not possible to complete the surgery.

But if L^m is a closed, smooth, simply connected π-manifold, the surgery problem $f \times 1_L: W \times L \to D^{4k} \times L$ does have a solution. To see this note first that if m is odd, the problem is odd dimensional so there are no obstructions to modifying $W \times L$ and $f \times 1_L$ by surgery to obtain a homotopy equivalence. If $m \equiv 0 \pmod 4$, the problem has an index obstruction given by the product of the index obstruction of the map f and the index of the manifold L, i.e., $I(f \times 1_L) = I(f) \cdot I(L)$. This product vanishes since L is a π-manifold. The formula follows from the multiplicativity of the index of a manifold. If $m \equiv 2 \pmod 4$ the problem has a Kervaire invariant obstruction given by the mod 2 product of the Kervaire invariant obstruction of f and the Euler characteristic of L, the formula arising from Sullivan's characterization of the Kervaire invariant obstruction [8]. Since L is a π-manifold, $\chi(L) = 0$; so $K(f \times 1_L)$ vanishes as well.

Now we change the surgery problem discussed above into a problem for closed manifolds. Let N be a smooth, closed, π-manifold of dimension $4k$. Take a small disk D^{4k} in N and form the connected sum $N \# \hat{W}$ using this disk and the disk attached to W to make $\hat{W}$. Define $1_N \# f\colon N \# \hat{W} \to N$ by the identity on N-int D^{4k} and f on W. Although $(1_N \# f) \times 1_L$ is not tangential, it can be surgered to a homotopy equivalence. This is because it is already a homotopy equivalence except on $W \times L$, where it is tangential; so it suffices to do surgery on $W \times L$ leaving the boundary fixed to make $N \# \hat{W} \times L$ homotopy equivalent to $N \times L$. We have already seen that this can be done. Summing up the discussion we have

PROPOSITION 1. *Suppose N^{4k} is a closed, smooth, orientable π-manifold and L^m is a closed, smooth, simply connected π-manifold. Then there is a manifold M^{4k+m}, homotopy equivalent to $N \times L$ obtained by surgery on $(1_N \# f) \times 1_L$.*

Notice that if $W_i^{4k} = \underbrace{W^{4k} \# \cdots \# W^{4k}}_{i}$, and we define $f_i\colon W_i \to D^{4k}$ the same way as we defined f, the above considerations also apply to W_i. The only difference is that W_i has index $8ri$. We shall denote the solution to the surgery problem using W_i by M_i^{4k+m}.

We also remark here that M, as a solution to a given surgery problem, is unique up to PL homeomorphism, but not not always up to diffeomorphism. This follows from Novikov's results [5]. Since we shall be primarily concerned with the topological type of such solutions, we shall ignore this ambiguity.

2. Properties of the surgery solution.

PROPOSITION 2. *The manifold M^{4k+m} obtained by surgery on*

$$(1_N \# f) \times 1_L \colon N \# \hat{W} \times L \to N \times L$$

is not a π-manifold.

Proof. After surgery we have a homotopy equivalence $g\colon M \to N \times L$ and a cobordism Z between M and $N \# W \times L$ together with a map $F\colon Z \to N \times L$ whose restriction is g on M and $(1_N \# f) \times 1_L$ on $N \# \hat{W} \times L$. If $*$ is a point of L, $(1_N \# f) \times 1_L$ is transverse regular with respect to $N \times *$. Change g by a small homotopy to make it transverse regular with respect to $N \times *$. Finally leaving $(1_N \# f) \times 1_L$ and g fixed, make F transverse regular with respect to $N \times *$ to obtain the oriented cobordism $F^{-1}(N \times *)$ between $N \# \hat{W}$ and

$$S = g^{-1}(N \times *) .$$

Because $N \# \hat{W}$ and S are oriented cobordant, $I(S) = I(N \# \hat{W}) \neq 0$. We have the usual equivalence of tangent and normal bundles

$$\tau(M) | S \cong \tau(S) \oplus \nu(S \subset M) .$$

Since f is transverse regular with respect to $N \times *$ and

$$\nu(N \times * \subset N \times L)$$

is trivial, $\nu(S \subset M)$ is trivial. Thus if $\nu(M) | S$ were stably trivial, $\tau(S)$ would be stably trivial, contradicting $I(S) \neq 0$. Therefore $\tau(M) | S$ is not stably trivial and consequently $\tau(M)$ is not stably trivial.

PROPOSITION 3. *M is not homeomorphic to $N \times L$.*

Proof. Suppose $h\colon M \to N \times L$ is a homeomorphism. Denote by $p_j(M)$ the j^{th} Pontrjagin class of M (i.e., of $\tau(M)$) and by $p_j(M; \mathbf{Q})$ the j^{th} rational Pontrjagin class of M. In the proof of Proposition 2 it was shown that M^{4k+m} contains a closed submanifold S of dimension $4k$ and index $8r$. If $i\colon S \to M$ is inclusion, the Hirzebruch index theorem implies

$$\begin{aligned} 8r &= \langle L_k(p_1(S), \cdots, p_k(S)), [S] \rangle \\ &= \langle L_k(i^* p_1(M), \cdots, i^* p_k(M)), [S] \rangle \\ &= \langle L_k(p_1(M), \cdots, p_k(M)), i_*[S] \rangle . \end{aligned}$$

Now we may replace $p_j(M)$ by $p_j(M; \mathbf{Q})$ since any torsion evaluated on the orientation class is zero. By the topological invariance of rational Pontrjagin classes, $p_j(M; \mathbf{Q}) = h^*(p_j(N \times L); \mathbf{Q})$; but

$$p_j(N \times L; \mathbf{Q}) = 0$$

for every j because $N \times L$ is *a* π-manifold. Therefore $p_j(M; \mathbf{Q}) = 0$

for every j, a contradiction.

Observe that Propositions 2 and 3 are likewise valid for the manifolds M_i, each M_i containing a closed submanifold S_i of dimension $4k$ and index $8ri$.

Now we are in a position to prove the central theorem of this paper.

THEOREM 1. *Suppose N is a smooth, closed, orientable π-manifold of dimension $4k (k \geqq 2)$ and L is a smooth, closed simply connected π-manifold. Then there is a countable sequence of smooth, closed manifolds $\{M_i\}$ having the following properties:* (1) *no M_i is a π-manifold,* (2) *each M_i is homotopy equivalent but not homeomorphic to $N \times L$,* (3) *M_i is not homeomorphic to M_j if $i \neq j$.*

Proof. The M_i's are the surgery solutions already described. Propositions 2 and 3 establish (1) and (2). It remains to prove (3). We do this by expanding the idea of the proof of Proposition 3.

Suppose there exists a homeomorphism $h: M_j \to M_i$ and $i \neq j$, say $i > j$. (For the rest of this paragraph $t = i, j$.) Let $g_t: M_t \to N \times L$ be a homotopy equivalence which is transverse regular with respect to $N \times *$ so that $g_t^{-1}(N \times *) = S_t$ where $I(S_t) = 8rt$. (We may assume that g_t is still the identity on $(N - \operatorname{int} D^{4k}) \times L$ since no surgery is done there.) Then by the index theorem,

$$\langle L_k(p_1(M_t; \mathbf{Q}), \cdots, p_k(M_t; \mathbf{Q})), [S_t] \rangle = I(S_t) .$$

To simplify notation we omit explicit reference to the inclusion maps $S_t \subset M_t$ and abbreviate $L_k(p_1(X; \mathbf{Q}), \cdots, p_k(X; \mathbf{Q}))$ by $L_k(X)$. Let $\bar{g}_t$ be a homotopy inverse for g_t. The idea is then to show that $g_i h \bar{g}_j$ does not behave properly on rational homology. We shall be referring to the following diagram for the rest of the proof:

$$\begin{tikzcd}
S_j \subset M_j \arrow[rr, "h"] \arrow[dr, shift left, "g_j"] & & M_i \supset S_i \arrow[dl, shift left, "g_i"] \\
& N \times L \arrow[ul, shift left, "\bar{g}_j"] \arrow[ur, shift left, "\bar{g}_i"] &
\end{tikzcd}$$

By the transverse regularity of g_t, it follows that

$$g_{t*}[S_t] = [N \times *] = [N] \otimes 1 \in H_{4k}(N \times L; \mathbf{Q}) ,$$

so $g_{j*} \bar{g}_{i*}[S_i] = [S_j]$. Thus

$$I(S_j) = \langle L_k(M_j), \bar{g}_{j*}g_{i*}[S_i]\rangle = \langle L_k(M_i), h_*\bar{g}_{j*}g_{i*}[S_i]\rangle$$

by the topological invariance of rational Pontrjagin classes.

Define a bundle ξ over $N \times L$ by $\bar{g}_i^*(\tau(M_i))$. This means that $\tau(M_i) = g_i^*(\xi)$. Since g_i is the identity on $N - \operatorname{int} D^{4k} \times L$ and

$$\tau(M_i) \mid N - \operatorname{int} D^{4k} \times L$$

is trivial, it follows that $\xi \mid N - \operatorname{int} D^{4k} \times L$ is trivial. Now if

$$i: N - \operatorname{int} D^{4k} \times L \to N \times L$$

is inclusion, then if $x \otimes y \in H_*(N \times L; \mathbf{Q})$ and $\dim x < 4k$, $x \otimes y \in$ image i_*, say $x \otimes y = i_*z$. Thus $\langle L_k(\xi), x \otimes y\rangle = \langle L_k(i^*\xi), z\rangle = 0$ since $i^*\xi$ is trivial. This shows that if $\gamma_{4k} \in H_{4k}(N \times L; \mathbf{Q})$, then $\langle L_k(\xi), \gamma_{4k}\rangle$ is given by the product of the coefficient of $[N] \otimes 1$ in γ_{4k} and

$$\langle L_k(\xi), [N] \otimes 1\rangle .$$

Using the preceding observation, we can compute the coefficient of $[N] \otimes 1$ in $(g_i h \bar{g}_j)_*[N] \otimes 1$ as follows.

$$\begin{aligned}\langle L_k(\xi), (g_i h \bar{g}_j)_*[N] \otimes 1\rangle &= \langle L_k(M_i), h_*\bar{g}_{j*}[N] \otimes 1\rangle \\ &= \langle L_k(M_i), h_*\bar{g}_{j*}g_{i*}[S_i]\rangle \\ &= I(S_j) = (j/i)I(S_i) .\end{aligned}$$

But

$$I(S_i) = \langle L_k(M_i), [S_i]\rangle = \langle L_k(\xi), g_{i*}[S_i]\rangle = \langle L_k(\xi), [N] \otimes 1\rangle .$$

Hence this coefficient is j/i which is not an integer since $i > j$. This contradicts the fact that any induced map on rational homology must send integral classes to integral classes.

3. An extension of the results. It has been pointed out to me that the results of this paper can be extended in the following way:

If M^n is a simply connected smooth manifold where n is odd and $H^{4k}(M; \mathbf{Q}) \neq 0$ or some $4k < n$, the Pontrjagin character shows that $\widetilde{KO}(M)$ is infinite. (See, for example, Hsiang [2].) Thus the kernel of $\widetilde{KO}(M) \to J(M)$ is infinite. It can be shown that the result of doing surgery on the elements of the kernel is a collection of smooth manifolds homotopy equivalent to M containing an infinite subset $\{M_i\}$ of mutually non-homeomorphic manifolds. The condition on the rational cohomology of M is also necessary for the manifolds $\{M_i\}$ exist.

Although the theorem described above considerably extends the class of manifolds to which the principal result applies, its proof requires methods of a deeper sort and the geometric simplicity is lost.

4. Applications. By an *H-manifold* we mean a closed, orientable topological manifold having the structure of an H-space.

THEOREM 2. *Suppose N^{4k} and L^m are smooth H-manifolds, N and L are π-manifolds, and L is simply connected. Then there exists a sequence of mutually nonhomeomorphic smooth H-manifolds $\{M_i\}$ satisfying* (1) *no M_i is a π-maifold,* (2) *each M_i is homotopy equivalent, but not homeomorphic to $N \times L$.*

Proof. This is immediate from Theorem 1 since the product of 2 H-manifolds is an H-manifold and any manifold homotopy equivalent to an H-manifold is itself an H-manifold.

Examples of manifolds nontangentially homotopy equivalent to Lie groups were known before surgery techniques were introduced; however all these were nonsimply connected. An example due to Milnor of a manifold homotopy equivalent to $S^1 \times S^3 \times S^7$ with a nonzero Pontrjagin class is quoted by Browder and Spanier [1].

The recent results of a A. Zabrodsky [9] and J. Stasheff [7] have produced new homotopy types of H-manifolds (other than compact Lie groups) to which Theorem 2 applies. However if we restrict ourselves to simply connected, compact Lie groups, we can obtain a stronger conclusion.

THEOREM 3. *Suppose N^{4k} and L^m are simply connected compact Lie groups ($k \geqq 2$). Then there is a countable sequence of mutually nonhomeomorphic H-manifolds $\{M_i\}$ satisfying* (1) *no M_i is a π-manifold,* (2) *each M_i is homotopy equivalent to $N \times L$ but not homeomorphic to any Lie group.*

Proof. Since Lie groups are π-manifolds, Theorem 1 applies. H. Scheerer has proved [6] that homotopy equivalent, compact, simply connected Lie groups are isomorphic; so if M_i were homeomorphic to any Lie group, it would be homeomorphic to $N \times L$, contradicting Theorem 1.

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