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ORDERED CYCLE LENGTHS IN A RANDOM PERMUTATION

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Let $x(t)$ denote the number of jumps occurring in the time interval $[0, t)$ and $v_k(t) = P\{x(t) = k\}$. The generating function of $v_k(t)$ is given by

$$\exp\{\lambda t[\phi(x) - 1]\},\ \phi(x) = \sum_{k=1}^{\infty} p_k x^k,\ \sum_{k=1}^{\infty} p_k = 1 \ .$$

Lay off to the right of the origin successive intervals of length z^j/j^α, $j = 1, 2, \cdots$. Explicitly the end points are

$$t_1(z) = 0$$

$$t_j(z) = \sum_{k=1}^{j-1} z^k/k^\alpha,\ j = 2, 3, \cdots, \alpha > 0 \ ,$$

and

$$t_\infty(z) = \sum_{k=1}^{\infty} z^k/k^\alpha \ .$$

Following Shepp and Lloyd L_r, the length of the rth longest cycle and S_r, the length of the rth shortest cycle have been defined for our choice of $x(t)$ and t_j, $j = 1, 2, \cdots$. This paper obtains the asymptotics for the mth moments of L_r and S_r suitably normalized by a new technique of generating functions. It is further shown that the results of Shepp and Lloyd are particular cases of these more general results.

Here we consider a problem involving a random permutation which is closely linked with the cycle structure of the permutation. Let S_n be the $n!$ permutation operators on n numbered places. Let $\alpha(\pi) = \{\alpha_1(\pi), \alpha_2(\pi), \cdots, \alpha_n(\pi)\}$ be the cycle class of $\pi \in S_n$. In this permutation π, there are $\alpha_1(\pi)$ cycles of length one, $\alpha_2(\pi)$ cycles of length two, etc. Usually the elements of S_n are assigned a probability $1/n!$ each. John Riordan has considered a model where he has assigned the probability

$$P\{\alpha_1 = a_1, \alpha_2 = a_2, \cdots, \alpha_n = a_n\} = \prod_{j=1}^{n} (1/j)^{a_j}/a_j! \text{ if } \sum_{j=1}^{n} ja_j = n, \tag{1.1}$$
$$= 0 \text{ otherwise} \ ,$$

for the cycle class $\alpha(\pi)$, the a's being nonnegative integers. Here α's would be independent if it were not for the condition $\sum ja_j = n$. Shepp and Lloyd has considered a sequence $\alpha = \{\alpha_1, \alpha_2, \cdots\}$ of mutually independent nonnegative integral valued random variables where for $j = 1, 2, \cdots$ the random variable α_j follows the Poisson distribution with mean z^j/j, $0 < z < 1$, z being same for all values of j. Accordingly

1.2
$$P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \cdots\} = (1 - z)z^{\sum_{j=1}^{\infty} ja_j} \prod_{j=1}^{\infty} (1/j)^{a_j}/a_j!, \quad a_j > 0, j = 1, 2, \cdots .$$

From this it can be seen that the probability distribution of the random variable $\nu(\alpha) = \sum_{j=1}^{\infty} j\alpha_j$ is

1.3
$$P\{\nu(\alpha) = n\} = (1 - z)z^n, n = 0, 1, 2, \cdots .$$

Also

1.4
$$P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \cdots | \nu(\alpha) = n\} = \prod_{j=1}^{n} (1/j)^{a_j}/a_j!, \sum_{j=1}^{\infty} ja_j = n$$
$$= 0 \text{ otherwise} .$$

Thus Shepp and Lloyd were able to recover 1.1 assumed in the model. In this paper, for the cycle class $\alpha(\pi)$ we have assigned the probability

1.5
$$P_z(\alpha_1 = a_1, \alpha_2 = a_2, \cdots, \alpha_n = a_n) = \mathrm{I}/\mathrm{II}, 0 < z < 1, \sum_{j=1}^{n} ja_j = n$$
$$= 0 \text{ otherwise} .$$

Here

1.6
$$I = \prod_{j=1}^{\infty} v_{a_j}(z^j/j^\alpha), \sum_{j=1}^{n} ja_j = n, a_{n+1} = a_{n+2} = \cdots = 0, (\sum_{j=1}^{\infty} ja_j = n)$$

where $v_{a_j}(z^j/j^\alpha)$ is the coefficient of x^{a_j} in $\exp\{\lambda(z^j/j^\alpha)[\phi(x) - 1]\}$,

1.7
$$\phi(x) = \sum_{k=1}^{\infty} p_k x^k \quad \text{and} \quad \sum_{k=1}^{\infty} p_k = 1 .$$

On detailed computation

1.8
$$v_{a_j}(z^j/j^\alpha) = e^{-\lambda z^j/j^\alpha} \sum_{n_1+2n_2+3n_3+\cdots=a_j} \frac{(p_1 z^j/j^\alpha)^{n_1}(p_2 z^j/j^\alpha)^{n_2}\cdots}{n_1!\, n_2! \cdots} .$$

In the special case when $\lambda = 1$, $p_1 = 1$, $p_2 = p_3 = \cdots = 0$ and $\alpha = 1$, $\exp\{\lambda(z^j/j^\alpha)[\phi(x) - 1]\}$ reduces to the generating function of the Poisson process with the time parameter equals to z^j/j, which has been considered by Shepp and Lloyd. Also the generating function of II which represents the distribution of $P\{\nu(\alpha) = n\}$, where for our choice of the sequence α_j's defined by 1.14

1.9
$$\nu(\alpha) = \sum_{j=1}^{\infty} j\alpha_j$$

is given by

1.10 $$\sum_{n=1}^{\infty} P\{\nu(\alpha) = n\}x^n = \prod_{j=1}^{\infty} \exp\{\lambda(z^j/j^\alpha)[\phi(x^j) - 1]\} .$$

On detailed computation we note that

1.11 $$P\{\nu(\alpha) = n\} = \exp\{-\lambda \sum_{j=1}^{\infty} z^j/j^\alpha\} \times \sum_{\substack{n_1+2n_2+3n_3+\cdots \\ +2(n_1'+2n_2'+\cdots) \\ +3(n_1''+2n_2''+\cdots) \\ +\cdots=n}} \left\{ \left[\frac{\left(\frac{p_1\lambda z}{1^\alpha}\right)^{n_1}\left(\frac{p_1\lambda z^2}{2^\alpha}\right)^{n_2}\cdots}{n_1!\, n_2! \cdots}\right] \times \left[\frac{\left(\frac{p_2\lambda z}{1^\alpha}\right)^{n_1'}\left(\frac{p_2\lambda z^2}{2^\alpha}\right)^{n_2'}\cdots}{n_1'!\, n_2'! \cdots}\right] \times \left[\frac{\left(\frac{p_3\lambda z}{1^\alpha}\right)^{n_1''}\left(\frac{p_3\lambda z^2}{2^\alpha}\right)^{n_2''}\cdots}{n_1''!\, n_2''! \cdots}\right] \times \cdots \right\} .$$

In particular when $\lambda = 1, \alpha = 1$ and $p_1 = 1, p_2 = p_3 = \cdots = 0$, the generating function of the distribution of 1.9 reduces to

1.12 $$\exp[-\sum (z^j/j) + \sum (x^j z^j/j)] = (1 - z)/(1 - zx) .$$

Hence

$$P\{\nu(\alpha) = n\} = (1 - z)z^n ,$$

which is in agreement with that considered by Shepp and Lloyd. In the special case mentioned above

1.13 $$\begin{aligned} \mathrm{I/II} &= \prod_{j=1}^{n} (1/j)^{a_j}/a_j! \qquad \text{if} \qquad \sum_{j=1}^{n} ja_j = n, \\ &= 0 \qquad \text{otherwise} . \end{aligned}$$

This is also in agreement with the model discussed by Shepp and Lloyd.

If we take $\alpha = (\alpha_1, \alpha_2, \cdots)$ to be a sequence of mutually independent nonnegative integral valued random variables where for $j = 1, 2, \cdots$

1.14 $$P_z\{\alpha_j = a_j\} = v_{a_j}(z^j/j^\alpha),\ a_j = 0, 1, 2, \cdots ,$$

by using the Borel-Cantelli lemma, we can easily show that $\nu(\alpha) = \sum_{j=1}^{\infty} j\alpha_j$ is finite with probability one. Hence the joint distribution $(\alpha_1, \alpha_2, \alpha_3, \cdots, \nu(\alpha))$ can be written as

1.15 $$\begin{aligned} P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \cdots, \nu(\alpha) = n\} &= \prod_{j=1}^{\infty} v_{a_j}(z^j/j^\alpha) \text{ if } \sum_{j=1}^{\infty} ja_j = n, \\ &= 0 \qquad \text{otherwise} . \end{aligned}$$

From this we can see that

1.16 $$P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \cdots, \mid \nu(\alpha) = n\} = \mathrm{I}/\mathrm{II}\ ,$$

which we have assumed for the model.

Shepp and Lloyd have considered a Poisson process which takes place on $T = \{-\infty < t < +\infty\}$ at unit rate. That is, for any interval of length $I \subset T$, the probability that p jumps occur in I is

$$\exp[-|I|]\,|I|^p/p!,\ p = 0, 1, 2, \cdots$$

independently of any conditions on the process on $T - I$. They have taken the following end points for the time intervals

1.17 $$\begin{aligned} t_1(z) &= 0, \\ t_j(z) &= \sum_{k=1}^{j-1} z^k/k,\ j = 2, 3, \cdots\ , \\ t_\infty(z) &= \sum_{k=1}^{\infty} z^k/k \quad = \log(1-z)^{-1}\ , \end{aligned}$$

so that the jth interval is

$$t_j(z) < t < t_{j+1}(z),\ j = 1, 2, \cdots .$$

They define $\lambda_z(t);\ -\infty < t < \infty$, to be a function whose value is 'j' on the jth interval, $j = 1, 2, \cdots$ and is zero if $t < 0$ or $t > t_\infty(z)$. Then for each $j = 1, 2, \cdots$ the interval $\{t; \lambda_z(t) = j\}$ has length z^j/j, the probability that a_j jumps of the Poisson process occur in this interval is

1.18 $$\exp(-z^j/j)\cdot(z^j/j)^{a_j}/a_j!,\ a_j = 0, 1, 2, \cdots$$

and that these various events for $j = 1, 2, \cdots$ are mutually independent. They have taken a sample function of the Poisson process, with jumps in the interval $[0, t_\infty(z))$, which are finite in number with probability one, occurring at times $\tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_\sigma$ (σ, random). They take the positive integers $\lambda_z(\tau_1) \leqq \lambda_z(\tau_2) \leqq \cdots \leqq \lambda_z(\tau_\sigma)$ as the lengths of the σ cycles of a permutation on $\nu = \sum_{s=1}^{\sigma} \lambda_z(\tau_s)$ places, and in this class S_ν, they choose a permutation at random with uniform distribution. For any given $r = 1, 2, \cdots$ let $S_r = S_r(\alpha)$ be the length of the rth shortest cycle in a permutation of the cycle class $\alpha \cdot S_r(\alpha) = 0$ if $\sum \alpha_j < r$. If the rth jump of the Poisson process occur at 't', then $S_r = \lambda_z(t)$ according to their model. Hence they have obtained the distribution of S_r. Similarly they have obtained the distribution of $L_r = L_r(\alpha)$, the length of the rth longest cycle. They have given asymptotics for the distribution and to all moments of the length of the rth longest and rth shortest cycles.

In this paper, instead of the Poisson process considered by Shepp and Lloyd, we consider a more general process which can have $k (k > 1)$

jumps at any moment. Let $x(t)$ denote the number of jumps in the interval $[0, t)$ and let

1.19 $$v_k(t) = P\{x(t) = k\} .$$

Let p_k be the probability of having k jumps at a chosen moment, if it is certain that jumps do occur generally at that moment. It has been shown in Khintchine that

1.20 $$F(t, x) = \sum_{k=0}^{\infty} v_k(t)x^k = \exp\{\lambda t[\phi(x) - 1]\} ,$$

where $\phi(x)$ is given by (1.7) and $\lambda > 0$. In our model, we take the end points of the time intervals to be

1.21 $$t_1(z) = 0$$
$$t_j(z) = \sum_{k=1}^{j-1} z^k/k^\alpha, \; j = 2, 3, \cdots, \alpha > 0 ,$$

and

$$t_\infty(z) = \sum_{k=1}^{\infty} z^k/k^\alpha .$$

Here the probability that L_r, the length of the rth longest cycle is 'j' is given by

1.22 $$\begin{aligned} P_z\{L_r = j\} &= \frac{\lambda}{\sum_{k=1}^{r} p_k} \int_{t_j}^{t_{j+1}} \left\{\sum_{k=1}^{r} p_k v_{r-k}(t_\infty - t)\right\} dt, \\ &= \frac{\lambda}{P_r} \int_{t_j}^{t_{j+1}} \left\{\sum_{k=1}^{r} p_k v_{r-k}(t_\infty - t)\right\} dt , \end{aligned}$$

where

$$P_r = \sum_{k=1}^{r} p_k .$$

Also the probability that S_r, the length of the rth shortest cycle is 'j' is given by

$$P_z\{S_r = j\} = \frac{\lambda}{P_r} \int_{t_j}^{t_{j+1}} \left\{\sum_{k=1}^{r} p_k v_{r-k}(t)\right\} dt .$$

Here we use the technique of generating functions to estimate the asymptotics of $E\{L_r\}^m$ and $E\{S_r\}^m$ suitably normalized in a way different from that used by Shepp and Lloyd. While they have considered the case where the jumps occur according to Poisson law, we have considered a more general system of which Poisson process is a special case. By assuming the Poisson law for jumps they were able to recover the model based on the uniform distribution. By assuming a more general law for jumps we obtain a generalised probability model for the cycle class of which that derived on the basis of the uniform distribution is a special case. Thus we have in this paper discussed a generalization of the one given by Shepp and Lloyd with the help of the new technique.

2. A lemma. We now prove a lemma which we use extensively.

LEMMA. *Let*

2.1 $$A(z, x) = \sum_{r=1}^{\infty} a_r(z)x^r ,$$

and

2.2 $$A(x) = \sum_{r=1}^{\infty} a_r x^r ,$$

with $a_r(z) > 0$, *satisfying*

2.3 $$\sum_{r=1}^{\infty} a_r(z) = c,\ 0 < z < 1 ,$$

c, a constant. Then for

2.4 $$a_r(z) \longrightarrow a_r,\ z \longrightarrow 1^- ,$$

it is necessary and sufficient that for $0 < x < 1$

2.5 $$A(z, x) \longrightarrow A(x),\ z \longrightarrow 1^- .$$

Proof of the lemma. First let us suppose that (2.4) holds. Then for fixed x, $(0 < x < 1)$ and ε, we can choose a number n_0 such that $\{x^{n_0}/(1 - x)\} < \varepsilon$. Then,

2.6 $$|A(z, x) - A(x)| < \sum_{r=0}^{n_0} |a_r(z) - a_r| x^r + 2c\varepsilon .$$

Now each term in the right hand side tends to zero. Hence the necessary part. Now suppose that (2.5) holds. Since $\{a_r(z)\}$ is bounded it is always possible to find a converging subsequence. If (2.4) is not true then we can extract two subsequences converging to two different sequences $\{a_r^*\}$ and $\{a_r^{**}\}$ and the corresponding subsequences of $\{A(z, x)\}$ would converge to $A^*(x) = \sum a_r^* x^r$ and $A^{**}(x) = \sum a_r^{**} x^r$ which contradicts the assumption that (2.5) holds. Hence $\{a_r^*\} = \{a_r^{**}\} = \{a_r\}$. This proves the sufficiency part.

3. The rth longest cycle. The mth raw moment of the rth longest cycle is

3.1 $$E_z\{L_r\}^m = \lambda \sum_{j=1}^{\infty} \frac{j^m}{P_r} \int_{t_j}^{t_{j+1}} \sum_{k=1}^{r} p_k v_{r-k}(t_\infty - t)\, dt .$$

Hence

3.2
$$\begin{aligned}\sum_{r=1}^{\infty} P_r x^{r-1} E_z\{L_r\}^m &= \lambda \sum_{r=1}^{\infty} x^{r-1} \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} \sum_{k=1}^{r} p_k v_{r-k}(t_\infty - t)dt, \\ &= \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} \sum_{r=1}^{\infty} x^{r-1} \Big\{\sum_{k=1}^{r} v_{r-k}(t_\infty - t) p_k\Big\} dt, \\ &= \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} e^{\lambda[\phi(x)-1](t_\infty - t)} \{\phi(x)/x\} dt .\end{aligned}$$

Let $F = F(\lambda)$ denotes the left hand side of (3.2) and $F' = F(\lambda s^{1-\alpha})$.

3.3
$$\begin{aligned}F' &= s^{1-\alpha}\lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} e^{\lambda s^{1-\alpha}[\phi(x)-1](t_\infty - t)} \{\phi(x)/x\} dt , \\ &= \sum_{r=1}^{\infty} P_r x^{r-1} E_z(L'_r)^m ,\end{aligned}$$

where L'_r is the same as L_r with λ replaced by $\lambda s^{1-\alpha}$.
Let us now consider some analytical preliminaries regarding $t_j(z)$. With $z = e^{-s}$, $0 < s < \infty$. We have

3.4
$$t_\infty(e^{-s}) - t_j(e^{-s}) = \sum_{k=j}^{\infty} \{e^{-ks}/k^\alpha\} .$$

In the interval $\{y\colon ks < y < (k+1)s\}$, we have

$$\frac{e^{-ks}}{k^\alpha s^\alpha} > \frac{e^{-y}}{y^\alpha} > \frac{e^{-(k+1)s}}{(k+1)^\alpha s^\alpha} ,$$

and

3.5
$$\frac{e^{-ks}s^{1-\alpha}}{k^\alpha} > \int_{ks}^{(k+1)s} \frac{e^{-y}}{y^\alpha} dy > \frac{s^{1-\alpha}e^{-(k+1)s}}{(k+1)^\alpha} .$$

Summing with respect to k, we have,

3.6
$$s^{1-\alpha} \sum_{k=j}^{\infty} (e^{-ks}/k^\alpha) > \int_{js}^{\infty} (e^{-y}/y^\alpha) dy .$$

Let

3.7
$$E(\theta) = \int_{\theta}^{\infty} (e^{-y}/y^\alpha) dy .$$

Then from (3.6) $E(js) < s^{1-\alpha} \sum_{k=j}^{\infty} e^{-ks}/k^\alpha$. Also

$$\int_{(j-1)s}^{\infty} (e^{-y}/y^\alpha) dy > s^{1-\alpha} \sum_{k=j}^{\infty} \{e^{-ks}/k^\alpha\} .$$

Combining the two

3.8
$$E(js) < s^{1-\alpha} \sum_{k=j}^{\infty} \{e^{-ks}/k^\alpha\} < E\{(j-1)s\} .$$

Now consider the equation

3.9 $$s^{1-\alpha} \sum_{k=j}^{\infty} \{e^{-ks}/k^{\alpha}\} = E(X) .$$

If $X_j(s)$ is the root of the equation (3.9), we have

3.10 and (i) $(j-1)s < X_j(s) < js$
(ii) $X_j(s)$ is unique .

In (3.3) put $E(\theta) = s^{1-\alpha}(t_\infty - t)$ so that

$$s^{1-\alpha}\, dt = \{e^{-\theta}/\theta^{\alpha}\} d\theta .$$

Hence

3.11 $$F' = \lambda \sum_{j=1}^{\infty} j^m \int_{X_j(s)}^{X_{j+1}(s)} \{\phi(x)/x\} \frac{e^{\lambda[\phi(x)-1]E(\theta)-\theta}}{\theta^{\alpha}} d\theta .$$

Let

$$\mu_j = \int_{X_j(s)}^{X_{j+1}(s)} d\mu(\theta) ,$$

where

3.12 $$d\mu(\theta) = \{e^{\lambda[\phi(x)-1]E(\theta)-\theta}/\theta^{\alpha}\} d\theta .$$

But

3.13 $$(j-1)s < X_j(s) < js \quad \text{and} \quad js < X_{j+1}(s) < (j+1)s .$$

This implies that

$$X_j(s) < js < X_{j+1}(s) .$$

Thus

$$s^m F' = \frac{\lambda\phi(x)}{x} \sum_{j=1}^{\infty} (js)^m \int_{X_j(s)}^{X_{j+1}(s)} d\mu(\theta) .$$

Now

3.14 $$\frac{\lambda\phi(x)}{x} \sum_{j=1}^{\infty} X_j^m(s)\mu_j \leqq F's^m \leqq \frac{\lambda\phi(x)}{x} \sum_{j=1}^{\infty} X_{j+1}^m(s)\mu_j .$$

Consider

3.15 $$\int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) = \sum_{j=1}^{\infty} \int_{X_j(s)}^{X_{j+1}(s)} \theta^m d\mu(\theta) .$$

We have

3.16 $$\sum_{j=1}^{\infty} X_j^m(s)\mu_j \leqq \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) \leqq \sum_{j=1}^{\infty} X_{j+1}^m(s)\mu_j .$$

i.e.,

$$I_1 \leqq I \leqq I_2 \quad \text{(say)} ,$$

where

$$I_1 = \sum_{j=1}^{\infty} X_j^m(s)\mu_j,\ I_2 = \sum_{j=1}^{\infty} X_{j+1}^m(s)\mu_j$$

and

$$I = \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) .$$

I_1 and I_2 are the Darboux sums for the Stieltjes integral based on the above meshes. Also $X_1(s) \to 0$ as $s \to 0^+$. Hence

3.17
$$\begin{aligned} s^m F' &\sim \{\phi(x)/x\}\lambda \int_0^{\infty} \theta^{m-\alpha} e^{\lambda[\phi(x)-1]E(\theta)-\theta} d\theta,\ s \to 0^+,\ m \geqq \alpha , \\ &\sim \lambda \int_0^{\infty} \theta^{m-\alpha} e^{-\theta} d\theta \sum_{r=1}^{\infty} x^{r-1}\Big\{\sum_{k=1}^{r} v_{r-k}[E(\theta)]p_k\Big\},\ s \to 0^+ . \end{aligned}$$

Now

3.18
$$\begin{aligned} s^m \sum_{r=1}^{\infty} P_r E_z(L_r')^m &= \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_j+1} dt = \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m (t_{j+1} - t_j) \\ &= \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m \{e^{-js}/j^{\alpha}\} = \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} \{e^{-js}/j^{\alpha-m}\} < \infty . \end{aligned}$$

Hence using the lemma

3.19
$$s^m P_r E_z(L_r')^m \sim \lambda \int_0^{\infty} \Big[\sum_{k=1}^{r} v_{r-k}[E(\theta)]p_k\Big] e^{-\theta}\theta^{m-\alpha} d\theta,\ s \to 0^+ .$$

Since $s \sim (1 - z)$,

$$(1 - z)^m E_z(L_r')^m \sim (\lambda/P_r) \int_0^{\infty} \Big[\sum_{k=1}^{r} v_{r-k}[E(\theta)]p_k\Big] e^{-\theta}\theta^{m-\alpha} d\theta,\ z \to 1^- .$$

Taking $\lambda = 1, \alpha = 1, p_1 = 1, p_2 = p_3 \cdots = 0$, we now have

3.20
$$\begin{aligned} (1 - z)^m E_z\{L_r\}^m &\sim \int_0^{\infty} v_{r-1}[E(\theta)] e^{-\theta}\theta^{m-1} d\theta,\ z \to 1^-, \\ &\sim \int_0^{\infty} e^{-E(\theta)-\theta}[E(\theta)]^{r-1}\{\theta^{m-1}/(r - 1)!\} d\theta,\ z \to 1^- . \end{aligned}$$

This is in agreement with Shepp and Lloyd.

4. The rth shortest cycle. Let S_r be the length of the rth shortest cycle. Then

4.1
$$P\{S_r = j\} = (\lambda/P_r) \int_{t_j}^{t_{j+1}} \sum_{k=1}^{r} p_k v_{r-k}(t) dt .$$

Let

$$F_1 = F_1(\lambda) = \sum_{r=1}^{\infty} P_r x^{r-1} E_z\{S_r\}^m .$$

Then

4.2
$$F_1 = \lambda \sum_{r=1}^{\infty} x^{r-1} \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} \sum_{k=1}^{r} p_k v_{r-k}(t)dt ,$$
$$= \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} e^{\lambda[\phi(x)-1]t}\{\phi(x)/x\}dt .$$

Also
$$F_1' = F_1(\lambda s^{1-\alpha}) = \sum P_r x^{r-1} E_z(S_r')^m ,$$

where S_r' is the same as S_r with λ replaced by $\lambda s^{1-\alpha}$. Put $(t_\infty - t)s^{1-\alpha} = E(\theta)$ in F_1'.

4.3
$$F_1' = \lambda \sum_{j=1}^{\infty} j^m \int_{X_j(s)}^{X_{j+1}(s)} \{\phi(x)/(x\theta^\alpha)\} e^{\lambda[s^{1-\alpha}t_\infty - E(\theta)][\phi(x)-1]-\theta} d\theta .$$

Let
$$\mu_j = \int_{X_j(s)}^{X_{j+1}(s)} d\mu(\theta) ,$$

where

4.4
$$d\mu(\theta) = \{\phi(x)/x\theta^\alpha\} e^{\lambda[s^{1-\alpha}t_\infty - E(\theta)][\phi(x)-1]-\theta} d\theta .$$

Hence

4.5
$$s^m F_1' = \lambda \sum_{j=1}^{\infty} (js)^m \int_{X_j(s)}^{X_{j+1}(s)} \{\phi(x)/x\theta^\alpha\} e^{\lambda[s^{1-\alpha}t_\infty - E(\theta)][\phi(x)-1]-\theta} d\theta .$$

Since $(j-1)s < X_j(s) < js < X_{j+1}(s) < (j+1)s$,

4.6
$$\lambda \sum_{j=1}^{\infty} X_j^m(s)\mu_j < F_1' s^m < \lambda \sum_{j=1}^{\infty} X_{j+1}^m(s)\mu_j .$$

Also
$$\sum_{j=1}^{\infty} X_j^m(s)\mu_j < \sum_{j=1}^{\infty} \int_{X_j(s)}^{X_{j+1}(s)} \theta^m d\mu(\theta) < \sum_{j=1}^{\infty} X_{j+1}^m(s)\mu_j .$$

That is

4.7
$$\sum_{j=1}^{\infty} X_j^m(s)\mu_j < \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) < \sum_{j=1}^{\infty} X_{j+1}^m(s)\mu_j .$$

Hence

4.8
$$s^m F_1' \sim \lambda \int_0^\infty \theta^{m-\alpha}\{\phi(x)/x\} e^{\lambda[s^{1-\alpha}t_\infty - E(\theta)][\phi(x)-1]-\theta} d\theta,\ s \to 0^+, \quad m \geqq \alpha .$$

Here also $s^m \sum_{r=1}^{\infty} P_r E_z(S_r')^m = s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m(t_{j+1} - t_j) < \infty$ {by (3.18)}. Thus as in 3.17 by equating the coefficient of x^{r-1} on both sides we can obtain $\lim_{s\to 0} s^m P_r E_z(S_r')^m$.

Now let us consider the particular case of the above when $p_1 = 1$, $p_2 = p_3 = \cdots = 0$ $\lambda = 1$ and $\alpha = 1$. Here

4.9
$$s^m \sum_{r=1}^{\infty} x^{r-1} E_z(S_r)^m \sim \int_0^\infty \theta^{m-1} e^{(x-1)[\log(1-z)^{-1}] - (x-1)E(\theta) - \theta} d\theta,\ z \to 1^-,$$
$$\sim s \int_0^\infty \theta^{m-1} e^{-x[E(\theta)+\log s] + E(\theta) - \theta} d\theta,\ s \to 0^+ .$$

Hence

$$s^{m-1}\sum_{r=1}^{\infty} x^{r-1}E_z(S_r)^m \sim e^{x\log(s^{-1})}\int_0^{\infty} e^{-xE(\theta)+E(\theta)-\theta}\theta^{m-1}d\theta .$$

So

4.10
$$\frac{(1-z)^{m-1}}{(m-1)!}\sum_{r=1}^{\infty} x^{r-1}E_z(S_r)^m \sim \frac{1}{(m-1)!}\times \left[\int_0^{\infty} e^{E(\theta)-\theta}\theta^{m-1}\sum_{r=1}^{\infty}\frac{[-xE(\theta)]^{r-1}}{(r-1)!}d\theta\right]\times\left[\sum_{r=1}^{\infty}\frac{[x\log(1-z)^{-1}]^{r-1}}{(r-1)!}\right].$$

Equating coefficient of x^{r-1} on both sides of 4.10

$$\frac{(1-z)^{m-1}}{(m-1)}E_z(S_r)^m \sim \frac{1}{(m-1)!}\sum_{p=0}^{r-1}\left[\{[\log(1-z)^{-1}]^p/p!\}\times\left\{\int_0^{\infty}\frac{[-E(\theta)]^{r-1-p}\theta^{m-1}e^{E(\theta)-\theta}}{(r-1-p)!}d\theta\right\}\right],\ s\to 0^+$$

4.11
$$\sim \sum_{p=0}^{r-1}(1/p!)[\log(1-z)^{-1}]^pK(r-1-p,m),\ s\to 0^+ ,$$

where

4.12
$$K(q,m)=\int_0^{\infty}\frac{\theta^{m-1}[-E(\theta)]^qe^{E(\theta)-\theta}}{(m-1)!\,q!}d\theta$$

which is in agreement with Shepp and Lloyd.

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