PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

TRIANGULAR MATRICES WITH THE ISOCLINAL PROPERTY

LEROY J. DERR

Consider the system V_n of $n \times n$, lower triangular matrices over the real numbers with the usual operations of addition, multiplication and scalar multiplication and with the additional property that $a_{i+1,j+1} = a_{i,j}$ (isoclinal). It is shown that V_n is a commutative vector algebra. The principal theorem (§ 3) establishes the existence of an algebraic mapping of V_n into a ring of rational functions. This mapping associates a special set of basis elements in V_n with the classically known Eulerian Polynomials.

Some properties of the space V_n are outlined in § 2. Section 4 gives an application of the main theorem to a problem which motivated this study, namely, the inversion of certain matrices in V_n for arbitrary dimension n. The matrices with first columns $[1^m, 2^m, \cdots, n^m]$, $m = 0, 1, 2, \cdots$, are considered in particular.

2. Properties.

2.1. *Nomenclature.* A matrix $A = \{a_{i,j}\}$ is called isoclinal if $a_{i+1,j+1} = a_{i,j}$ for all values of the indices permitted. Further we designate by V_n the class of $n \times n$ lower-triangular, isoclinal (L.T.I.) matrices (over the reals).

REMARK. The isoclinal property has appeared in studies of commutativity, under other names; for example see [4].

THEOREM 2.2. *The class V_n is a commutative sub-ring of matrices. Further, if $A \in V_n$ is nonsingular then $A^{-1} \in V_n$.*

Proof. A simple computation using the L.T.I. property will show multiplicative closure. Now, for $A, B \in V_n$ let $\{a_i\}$, $\{b_i\}$ be the elements of their first columns; these clearly define the matrices. The first column of AB is given by the Cauchy Product formula $\sum_{j=1}^{k}, a_j b_{k-j+1}$ for $k = 1, 2, \cdots, n$, which is commutative. Finally, if $A \in V_n$ is nonsingular then its diagonal element $a_1 \neq 0$ and the system $a_1 x_1 = 1$, $\sum_{j=1}^{k}, a_j x_{k-j+1} = 0$ is solvable. Hence $X \in V_n$ and $X = A^{-1}$.

The algebra of V_n is closely allied to that of the polynomials over the reals, $P(Y)$. Let $A \in V_n$ be given by its first column $\{a_i\}$. Define ϕ_n: $V_n \to P(Y)$ as the injection, $\phi_n(A) = \sum_{j=1}^{n}, a_j Y^{j-1}$ and let π_n: $P(Y) \to V_n$ be the projection. We then have:

COROLLARY 2.3.

(i) *π_n is a ring homomorphism onto, with kernel the principal ideal generated by Y^n.*

(ii) *$\pi_n\phi_n$ is the identity and $\pi_n\{\phi_n(A)\phi_n(B)\} = AB$.*

Finally we note the useful operating rule for L.T.I. matrices that the product Ax, where x is a vector, is equivalent to AX where X is the L.T.I. matrix with first column x.

3. A Mapping of V_n by means of Eulerian Polynomials.

3.1. *Definitions and Nomenclature.* (i). The Eulerian Polynomials $A_m(\lambda)$ may be defined recursively, with $A_0(\lambda) = 1$, by:

$$A_{m+1}(\lambda) = (1 + m\lambda)A_m(\lambda) + \lambda(1 - \lambda)A'_m(\lambda) .$$

(ii) Let $M_{m,n} \in V_n$ be defined, (giving the matrices' first columns), by:

$$M_{m,n} = (1^m, 2^m, \cdots, n^m) \quad \text{for} \quad m = 0, 1, 2, \cdots .$$

(iii) Let $M_m(\lambda) = \sum_{p=1}^{\infty} p^m \cdot \lambda^{p-1}$, for $|\lambda| < 1$ and $m = 0, 1, \cdots$.

(iv) Let $R = \{P(\lambda)/Q(\lambda)\}$ be the sub-ring of rational functions such that $Q(0) \neq 0$.

3.2. *Assertion.* (i) The matrices $M_{m,n}$ constitute a basis for V_n, $m = 0, 1, \cdots, n - 1$.

(ii) $M_m(\lambda) = A_m(\lambda)/(1 - \lambda)^{m+1} \in R$.

The second part of the assertion may be easily proved by noting the recursion $M_{m+1}(\lambda) = d\{\lambda M_m(\lambda)\}/d\lambda$. The Eulerian Polynomials and rational functions closely related to the $M_m(\lambda)$ were used by Frobenius [2] in studies of Bernoulli numbers; a further exposition of their properties has been given by Carlitz [1] and they have been used by Riordan [3] in combinatorial analysis. The inversion of the matrices $M_{m,n}$ was the author's original problem and will be discussed in the next section. Now, using the above notations and definitions, we give the following algebraic mapping theorem.

THEOREM 3.3. *In the following diagram:*

$$V_n \xrightarrow{f_n} R \xrightarrow{h_n} R/\langle\lambda^n\rangle \xrightarrow{j_n} V_n$$

f_n is defined by identifying the basis elements of V_n, $f_n(M_{m,n}) = M_m(\lambda) \in R$. h_n is the natural homomorphism with kernel, $K(h_n)$, the principal ideal generated by λ^n. Then, there exists a ring isomorphism

j_n such that $j_n h_n(M_m(\lambda)) = M_{m,n}$.

Proof. We first note that an element γ of the ring $R/\langle\lambda^n\rangle$ has a unique antecedent in R of the form $\sum_{p=1}^n, a_p\lambda^{p-1}$. This enables us immediately to define j_n as an additive isomorphism onto by $j_n(\gamma) = (a_1, a_2, \cdots, a_n) \in V_n$. The product of two elements in $R/\langle\lambda^n\rangle$ can be expressed as $\sum_{p=1}^n, c_p\lambda^{p-1} + K(h_n)$ where the c_p are formed by Cauchy Products of the unique antecedents. This gives a ring isomorphism since the multiplication in V_n is also Cauchy Product, truncated to n components.

The conclusion $j_n h_n(M_m(\lambda)) = j_n h_n\{\sum_{p=1}^n, p^m\lambda^{p-1}\} = M_{m,n}$ follows at once by noting $M_m(\lambda) = \sum_{p=1}^n, p^m\lambda^{p-1} + \sum_{p=n+1}^\infty, p^m\lambda^{p-1}$. Other immediate consequences are:

COROLLARY 3.4. (i) *f_n is one-to-one and $j_n h_n f_n$ is the identity.* (ii) $j_n h_n\{f_n(A)\cdot f_n(B)\} = AB$.

4. Application.

By making use of the previous theorem:

$$M_{m,n}^{-1} = j_n h_n\{(1-\lambda)^{m+1}\}\cdot j_n h_n\{1/A_m(\lambda)\} = BC^{-1}\ .$$

The matrix B is given by its first column $(b_1, \cdots, b_n)$ where $b_i = (-1)^{i-1}\binom{m+1}{i-1}$ if $i \leqq m+2$ and $b_i = 0$ if $i > m+2$. The nonzero components for $C \in V_n$ are also finite in extent, being the coefficients of the Eulerian Polynomial $A_m(\lambda)$. These are known explicitly: $A(m, k) = \sum_{j=0}^k, (-1)^{k-j}(j+1)^m\binom{m+1}{k-j}, k = 0, 1, \cdots, m-1$. The problem is then reduced to finding C^{-1} which may be expressed in terms of a recursion on the $A(m, k)$. For $m = 0, 1, 2$ the solutions are trivial. For $m = 3$ the n^{th} component, c_n, of C^{-1} is $c_n = U_n(-2)$ (Chebyshev polynomials of the second kind). These are readily given in explicit form.

REFERENCES

1. L. Carlitz, *Eulerian numbers and polynomials*, Math. Mag. **32** (1959), 247-260.
2. G. Frobenius, *Uber die Bernoullischen und die Eulerschen Polynome*, Sitzungsberichte der Preussische Akademie der Wissenschaften (1910), 809-847.
3. J. Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958.
4. D. Suprunenko and R. Tyshkevich, *Commutative Matrices*, Academic Press, New York 1968.

Received July 22, 1970

LOUISIANA STATE UNIVERSITY IN NEW ORLEANS