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VON NEUMANN ALGEBRAS GENERATED BY OPERATORS SIMILAR TO NORMAL OPERATORS

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A normal operator generates an abelian von Neumann algebra. However, an operator which is similar to a normal operator may generate a von Neumann algebra which is not even type I. In fact, it is shown that if $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space and $\mathscr{A}$ has no type II finite summand, then $\mathscr{A}$ has a generator which is similar to a self-adjoint and $\mathscr{A}$ has a generator which is similar to a unitary. The restriction that $\mathscr{A}$ have no type II finite summand can be removed provided that it is assumed that every type II finite von Neumann algebra has a single generator.

Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{A}$ be a von Neumann algebra on $\mathscr{H}$. $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$. For $n \geqq 2$, let $M_n(\mathscr{A})$ denote the von Neumann algebra of $n \times n$ matrices with entries in $\mathscr{A}$. If T is a bounded operator, the $\mathscr{R}(T)$ is the von Neumann algebra generated by T.

We begin with some lemmas.

LEMMA 1. *Let $\mathscr{A} = \mathscr{R}(C)$ and suppose $n \geqq 3$. Let $\{\lambda_k\}_{k=1}^{n}$ and $\{a_k\}_{k=1}^{n-1}$ be sequences of complex numbers such that the λ_k are distinct, each $a_k \neq 0$, and $\|(\lambda_1 - \lambda_2)C\| \leqq |a_1 a_2|$. Define $A = (A_{i,j})_{i,j=1}^{n} \in M_n(\mathscr{A})$ by $A_{k,k} = \lambda_k I$, $A_{k+1,k} = a_k I$, $A_{3,1} = C$,* and *$A_{i,j} = 0$ otherwise. Define $B = (B_{i,j})_{i,j=1}^{n} \in M_n(\mathscr{A})$ by $B_{k,k} = \lambda_k I$ and $B_{i,j} = 0$ if $i \neq j$. Then A and B are similar, and $\mathscr{R}(A) = M_n(\mathscr{A})$.*

Proof. It follows from [11, Lemma 1] that $\mathscr{R}(A) = M_n(\mathscr{A})$. To show that A and B are similar we need only that the λ_k are distinct. We must find an invertible operator S such that $AS = SB$. Such an S of the form $S = I + N$, where N is lower triangular and nilpotent, can be computed easily. Merely perform the matrix multiplications and solve for the entries of S. We omit the details.

REMARK 1. If the operator $S = I + N$ in Lemma 1 is computed, we see that we can make the entries of N small by choosing $\|C\|$, $|a_1|, |a_2|, \cdots, |a_{n-1}|$ suitably small. Hence we can suppose that $\|N\| < 1/2$. Then $\|S| = \|I + N\| < 3/2$ and $\|S^{-1}\| = \|I - N + N^2 - \cdots \pm N^{n-1}\| < 2$. Note also that by choosing $\|C\|, |a_1|, |a_2|, \cdots, |a_{n-1}|$ suitably, we can assume that $\|A\| \leqq \|B\| + 1$.

The following is a corollary of Lemma 1.

COROLLARY 1. *If $\mathscr{A}$ is a properly infinite von Neumann algebra on $\mathscr{H}$, then $\mathscr{A}$ has a generator which is similar to a self-adjoint operator.*

Proof. If $\mathscr{A}$ is properly infinite, then it is well-known that $\mathscr{A}$ is *-isomorphic to $M_3(\mathscr{A})$. $\mathscr{A}$ has a single generator C by [10]. Construct a generator A of $M_3(\mathscr{A})$ as in Lemma 1, with λ_1, λ_2, and λ_3 real. Then A is similar to self-adjoint operator by Lemma 1. (Another easy proof of Corollary 1 can be deduced from methods in the proof of Corollary 1 in [1].)

It has been shown that if $\mathscr{A}$ is properly infinite, then $\mathscr{A}$ is generated by three projections [9] and by two idempotents [4]. A related result is

COROLLARY 2. *If $\mathscr{A}$ is a properly infinite von Neumann algebra on $\mathscr{H}$, then $\mathscr{A}$ is generated by three commuting idempotents.*

Proof. If A is the generator of $\mathscr{A}$ constructed in Corollary 1, let E be the (idempotent valued) spectral measure of A. Then $E(\lambda_1)$, $E(\lambda_2)$, and $E(\lambda_3)$ are the required commuting idempotents.

Let $\sigma(C)$ denote the spectrum of the operator C.

LEMMA 2. *Let $\mathscr{A} = \mathscr{R}(C)$. Let*

$$A = \begin{bmatrix} C & 0 \\ aI & \lambda I \end{bmatrix}, \qquad B = \begin{bmatrix} C & 0 \\ 0 & \lambda I \end{bmatrix},$$

where $a \neq 0$ and $\lambda \notin \sigma(C)$. Then A is similar to B, and $\mathscr{R}(A) = M_2(\mathscr{A})$.

Proof. A routine computation shows that

$$\mathscr{R}(A)' = \left\{ \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} : \quad T \in \mathscr{A}' \right\}.$$

It follows that $\mathscr{R}(A) = \mathscr{R}(A)'' = M_2(\mathscr{A})$. Let

$$S = \begin{bmatrix} I & 0 \\ a(C - \lambda I)^{-1} & I \end{bmatrix}.$$

Then S is invertible and $AS = SB$.

LEMMA 3. *Let $\{A_k\}_{k=0}^{\infty}$ be a uniformly bounded sequence of operators. Suppose that the A_k have pairwise disjoint spectra. Then*

$$\mathscr{R}\left(\sum_{k=0}^{\infty} \oplus A_k\right) = \sum_{k=0}^{\infty} \oplus \mathscr{R}(A_k) .$$

Proof. The proof given here is due essentially to Rosenthal [8, Th. 3]. (See also [3, Lemma].) Let $A = \sum_{k=0}^{\infty} \oplus A_k$. Suppose $C = (C_{i\,j})_{i,j=0}^{\infty}$ commutes with A. Then

$$C_{i,j}A_j = A_iC_{i,j} \quad \text{for all } i, j .$$

If $i \neq j$, then $\sigma(A_i)$ and $\sigma(A_j)$ are disjoint, so by a theorem of Rosenblum [7], $C_{i\,j} = 0$. It follows that $\mathscr{R}(A)' = \sum_{k=0}^{\infty} \oplus \mathscr{R}(A_k)'$, so that $\mathscr{R}(A) = \mathscr{R}(A)'' = \sum_{k=1}^{\infty} \oplus \mathscr{R}(A_k)$.

THEOREM 1. *If $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space such that $\mathscr{A}$ has no type II finite summand, then $\mathscr{A}$ has a generator which is similar to a self-adjoint operator.*

Proof. Write $\mathscr{A} = \sum_{n=0}^{\infty} \oplus \mathscr{A}_n$, where $\mathscr{A}_0$ is properly infinite and for each $n \geqq 1$, $\mathscr{A}_n$ is an n-homogeneous type I summand (see [2]). (Note that some of these summands may be absent.) Let $\{I_n\}_{n=0}^{\infty}$ be a pairwise disjoint sequence of nonempty subintervals of $[0, 1]$.

By Corollary 1, we can choose A_0 and an invertible operator S_0 such that $\mathscr{R}(A_0) = \mathscr{A}_0$, $S_0A_0S_0^{-1}$ is self-adjoint, and $\sigma(A_0) \subset I_0$.

For each $n \geqq 1$, $\mathscr{A}_n$ is *-isomorphic to $M_n(\mathscr{C}_n)$, where $\mathscr{C}_n$ is the center of $\mathscr{A}_n$ (see [2]). $\mathscr{C}_n$ is abelian, so $\mathscr{C}_n$ has a self-adjoint generator by [5]. Let A_1 be a self-adjoint generator of $\mathscr{A}_1 = \mathscr{C}_1$. By translating and scaling, if necessary, we can assume $\sigma(A_1) \subset I_1$. Let S_1 be the identity in $\mathscr{A}_1$.

Let C be a self-adjoint generator of $\mathscr{C}_2$ with $\sigma(C)$ properly contained in I_2. Let $\lambda \in I_2$ with $\lambda \notin \sigma(C)$. Let $a \neq 0$ and let

$$A_2 = \begin{bmatrix} C & 0 \\ aI & \lambda I \end{bmatrix}.$$

Then by Lemma 2, $\mathscr{R}(A_2) = \mathscr{A}_2$ and for some invertible S_2, $S_2A_2S_2^{-1}$ is self-adjoint. Also, $\sigma(A_2) = \sigma(C) \cup \{\lambda\} \subset I_2$.

For $n \geqq 3$, use Lemma 1 to construct A_n and an invertible S_n such that $\mathscr{R}(A_n) = \mathscr{A}_n$, $S_nA_nS_n^{-1}$ is self-adjoint, and $\sigma(A_n) \subset I_n$. Moreover by Remark 1, we can suppose that the sequences $\{A_n\}$, $\{S_n\}$, and $\{S_n^{-1}\}$ are uniformly bounded.

Let $A = \sum_{n=0}^{\infty} \oplus A_n$, and let $S = \sum_{n=0}^{\infty} S_n$. Then A and S are bounded operators, S is invertible, and SAS^{-1} is self-adjoint. Finally $\mathscr{R}(A) = \sum_{n=0}^{\infty} \oplus A_n$ by Lemma 3.

It has long been conjectured that every von Neumann algebra on a separable Hilbert space has a single generator. Results in [6] and

[10] reduce the proof of the conjecture to showing that (S) Every type II finite von Neumann algebra on a separable Hilbert space has single generator. (See [4] for a partial solution to this conjecture.)

THEOREM 2. *If (S) is true and $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space, then $\mathscr{A}$ has a generator which is similar to a self-adjoint operator.*

Proof. Write $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$, where $\mathscr{A}_1$ has no type II finite summand and $\mathscr{A}_2$ is type II finite. By Theorem 1, $\mathscr{A}_1$ has a generator A_1 which is similar to a self-adjoint operator. Construct a generator of $\mathscr{A}_2$ as follows: Choose a projection $E \in \mathscr{A}_2$ such that $\mathscr{A}_2$ is spatially *-isomorphic to $M_4(E\mathscr{A}_2E)$. $E\mathscr{A}_2E$ is type II finite, so $E\mathscr{A}_2E$ has a single generator by assumption. Now use Lemma 1 to construct a generator A_2 of $\mathscr{A}_2$ which is similar to a self-adjoint and such that $\sigma(A_1)$ and $\sigma(A_2)$ are disjoint. Then $A_1 \oplus A_2$ is similar to a self-adjoint operator, and $\mathscr{R}(A_1 \oplus A_2) = \mathscr{A}_1 \oplus \mathscr{A}_2$.

We now indicate briefly how the previous results can be obtained with "similar to a self-adjoint" replaced by "similar to a unitary,"

COROLLARY 1′. *If $\mathscr{A}$ is a properly infinite von Neumann algebra on $\mathscr{H}$, then $\mathscr{A}$ has a generator which is similar to a unitary operator.*

The proof is the proof of Corollary 1, except that λ_1, λ_2, and λ_3 must be chosen on the unit circle. (See [1] for another proof.)

THEOREM 1′. *If $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space such that $\mathscr{A}$ has no type II finite summand, then $\mathscr{A}$ has a generator which is similar to a unitary operator.*

Proof. Proceed as in the proof of Theorem 1. Write $\mathscr{A} = \sum_{n=0}^{\infty} \oplus \mathscr{A}_n$. Use Lemmas 1 and 2 and Corollary 1′ to construct generators A_n of the $\mathscr{A}_n$ which have pairwise disjoint spectra on the unit circle. Then each A_n will be similar to a unitary operator. To handle the summands $\mathscr{A}_1$ and $\mathscr{A}_2$, we need the following: If C is a self-adjoint generator of $\mathscr{C}$, then e^{iC} is a unitary generator of $\mathscr{C}$ and $\sigma(e^{iC}) = \{e^{i\lambda} \colon \lambda \in \sigma(C)\}$. The rest of the proof is clear.

Finally we have

THEOREM 2′. *If (S) is true and $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space, then $\mathscr{A}$ has a generator which is similar to a unitary operator.*

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