PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 2, 1971

GROUP RINGS SATISFYING A POLYNOMIAL IDENTITY II

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In an earlier paper we obtained necessary and sufficient conditions for the group ring $K[G]$ to satisfy a polynomial identity. In this paper we obtain similar conditions for a twisted group ring $K^t[G]$ to satisfy a polynomial identity. We also consider the possibility of $K[G]$ having a polynomial part.

1. Twisted group rings. Let K be a field and let G be a (not necessarily finite) group. We let $K^t[G]$ denote a twisted group ring of G over K. That is $K^t[G]$ is an associative K-algebra with basis $\{\bar{x} \mid x \in G\}$ and with multiplication defined by

$$\bar{x}\bar{y} = \gamma(x, y)\overline{xy}\ , \qquad \gamma(x, y) \in K - \{0\}\ .$$

The associativity condition is equivalent to $\bar{x}(\bar{y}\bar{z}) = (\bar{x}\bar{y})\bar{z}$ for all $x, y, z \in G$ and this is equivalent to

$$\gamma(x, yz)\gamma(y, z) = \gamma(x, y)\gamma(xy, z)\ .$$

We call the function $\gamma\colon G \times G \to K - \{0\}$ the factor system of $K^t[G]$. If $\gamma(x, y) = 1$ for all $x, y \in G$ then $K^t[G]$ is in fact the ordinary group ring $K[G]$. In this section we offer necessary and sufficient conditions for $K^t[G]$ to satisfy a polynomial identity. The proof follows the one for $K[G]$ given in [3] and we only indicate the suitable modifications needed. The following is Lemma 1.1 of [2].

LEMMA 1.1. *If $x \in G$, then in $K^t[G]$ we have*

(i) $1 = \gamma(1, 1)^{-1}\bar{1}$

(ii) $\bar{x}^{-1} = \gamma(x, x^{-1})^{-1}\gamma(1, 1)^{-1}\,\overline{x^{-1}}$
$\qquad = \gamma(x^{-1}, x)^{-1}\gamma(1, 1)^{-1}\,\overline{x^{-1}}$.

PROPOSITION 1.2. *Suppose $K^t[G]$ satisfies a polynomial identity of degree n and set $k = (n\,!)^2$. Then G has a characteristic subgroup G_0 such that $[G\colon G_0] \leqq (k + 1)\,!$ and such that for all $x \in G_0$*

$$[G\colon C_G(x)] \leqq k^{4(k+1)!}\ .$$

Proof. This is the twisted analog of Corollary 3.5 of [3]. We consider § 3 of [3] and observe that each of the prerequisite results for that corollary also has a twisted analog.

First Lemma 3.1 of [3] holds for $K^t[G]$ with no change in the proof. Of course x must be replaced by $\bar{x}$ in the formula

$$\alpha_1\bar{x}\beta_1 + \alpha_2\bar{x}\beta_2 + \cdots + \alpha_t\bar{x}\beta_t = \bar{x}\gamma \, .$$

Second Theorem 3.4 of [3] also holds for $K^t[G]$ with no change in its statement. The proof is modified just slightly so that the inductive result to be proved is as follows. For each $x_j, x_{j+1}, \cdots, x_n \in G$, then either $f_j(\bar{x}_j, \bar{x}_{j+1}, \cdots, \bar{x}_n) = 0$ or for some $\mu \in \mathscr{M}_j$, $\mu(\bar{x}_j, \bar{x}_{j+1}, \cdots, \bar{x}_n) = a\,\bar{y}$ for some $a \in K - \{0\}$, $y \in \Delta_k(G)$. Then replacing x's suitably by $\bar{x}$'s the proof carries through as before. Finally Corollary 3.5 of [3] holds for $K^t[G]$ since it is just a group theoretic consequence of Theorem 3.4 of [3].

Let $K^t[G]$ be a twisted group ring and let H be a subgroup of G. Then by $K^t[H]$ we mean that twisted group ring of H which is naturally contained in $K^t[G]$. Let $JK^t[G]$ denote the Jacobson radical of $K^t[G]$.

PROPOSITION 1.3. *Suppose $K^t[G]$ satisfies a polynomial identity of degree n and suppose further that G' is finite and $K^t[G']$ is central in $K^t[G]$. Then G has a subgroup $Z \supseteqq G'$ such that*

$$[G\colon Z] \leqq (n/2)^{2|G'|}$$

with $K^t[Z]/(JK^t[G'] \cdot K^t[Z])$ commutative.

Proof. Since $K^t[G']$ is commutative, $JK^t[G']$ is the intersection of the maximal two-sided ideals of $K^t[G']$. Moreover $K^t[G']/JK^t[G']$ is a finite dimensional semisimple algebra and hence it has at most

$$\dim_K K^t[G']/JK^t[G'] \leqq |G'|$$

maximal two-sided ideals. Thus we may write

$$JK^t[G'] = \bigcap\nolimits_1^m I_i, \; m \leqq |G'|$$

where I_i is a maximal two-sided ideal of $K^t[G']$.

Fix a subscript i. Then $K^t[G']/I_i = F_i$, some finite field extension of K. Now $K^t[G']$ is central in $K^t[G]$, so $I_i \cdot K^t[G]$ is an ideal in $K^t[G]$. It is now easy to see that $K^t[G]/(I_i \cdot K^t[G])$ is an F_i-algebra with a basis consisting of the images of coset representatives for G' in G. Thus clearly $K^t[G]/(I_i \cdot K^t[G])$ is isomorphic to some twisted group ring $F_i^{t_i}[G/G']$, and this twisted group ring inherits the polynomial identity satisfied by $K^t[G]$. Thus by Proposition 1.4 of [2], G/G' has a subgroup $\bar{Z}_i$ with $[G/G'\colon \bar{Z}_i] \leqq (n/2)^2$ and with $F_i^{t_i}[Z_i]$ central in $F_i^{t_i}[G/G']$. Let Z_i be the complete inverse image of $\bar{Z}_i$ in G. Then $Z_i \supseteq G'$, $[G: Z_i] \leqq (n/2)^2$ and for all $\alpha, \beta \in K^t[Z_i]$ we have $\alpha\beta - \beta\alpha \in I_i \cdot K^t[G]$.

Set $Z = \bigcap_1^m Z_i$. Then

$$[G: Z] \leqq \Pi_1^m [G: Z_i] \leqq (n/2)^{2m} \leqq (n/2)^{2|G'|} .$$

Moreover for all $\alpha, \beta \in K^t[Z]$ we have

$$\alpha\beta - \beta\alpha \in \bigcap_1^m I_i \cdot K^t[G] = JK^t[G'] \cdot K^t[G]$$

since $K^t[G]$ is free over $K^t[G']$. Hence since $K^t[G]$ is free over $K^t[Z]$ we have

$$\alpha\beta - \beta\alpha \in K^t[Z] \cap (JK^t[G'] \cdot K^t[G]) = JK^t[G'] \cdot K^t[Z]$$

and the result follows.

We now come to our main result on twisted group rings satisfying a polynomial identity.

THEOREM 1.4. *Let $K^t[G]$ be a twisted group ring of G over K. Let $G \supseteq A \supseteq B$ be subgroups of G with B finite and central in A and with $K^t[A]/(JK^t[B] \cdot K^t[A])$ commutative.*

(i) *If $[G: A] < \infty$ then $K^t[G]$ satisfies a polynomial identity of degree $n = 2[G: A] \cdot |B|$.*

(ii) *If $K^t[G]$ satisfies a polynomial identity of degree n, then there exists suitable A and B with $[G: A] \cdot |B|$ bounded by some fixed function of n.*

Proof. The proof of (i) is identical to the proof of Theorem 1.3 (i) of [3]. Observe that $JK^t[B] \cdot K^t[A] = K^t[A] \cdot JK^t[B]$ is an ideal of $K^t[A]$ by Lemma 1.2 of [1].

We now consider part (ii). Let $K^t[G]$ satisfy a polynomial identity of degree n. Set

$$a = a(n) = (n\,!)^2, \qquad b = b(n) = a^{4(a+1)!} .$$

Then by Proposition 1.2 G has a subgroup G_0 with

$$[G: G_0] \leqq (a+1)\,!, \qquad G_0 = \Delta_b(G_0)$$

where Δ_k is defined in [3].

Set

$$c = c(n) = (b^4)^{b4}, \qquad d = d(n) = (n/2)^{2c} .$$

Then by Theorem 4.4 of [3], $|G_0'| \leqq c$. Let $G_1 = C_{G_0}(G_0')$. Then $G_1' \subseteq G_0'$ so G_1' is a finite central subgroup of G_1. Moreover

$$|G_1'| \leqq c, \qquad [G_0: G_1] \leqq c\,! .$$

Let $x \in G_1$. Then conjugation by $\bar{x}$ induces an automorphism of $K^t[G_1']$. Moreover since G_1' is central in G_1 we have

$$\bar{x}^{-1}\bar{y}\bar{x} = \lambda_x(y)\bar{y}$$

for all $y \in G_1'$. It follows easily that λ_x is a linear character of G_1' into K, that is $\lambda_x \in \operatorname{Hom}(G_1', K - \{0\})$. In addition, it follows easily that the map $x \to \lambda_x$ is in fact a group homomorphism

$$G_1 \longrightarrow \operatorname{Hom}(G_1', K - \{0\}) .$$

Let G_2 denote the kernel of this homomorphism. Then

$$[G_1 \colon G_2] \leqq |\operatorname{Hom}(G_1', K - \{0\})| \leqq |G_1'| \leqq c .$$

Set $B = G_2'$. Then $B \subseteqq G_1'$ so $|B| \leqq c$ and $K^t[B]$ is central in $K^t[G_2]$. By Proposition 1.3, G_2 has a subgroup $A \supseteqq B$ with

$$[G_2 \colon A] \leqq (n/2)^{2|B|} \leqq d$$

and with $K^t[A]/(JK^t[B] \cdot K^t[A])$ commutative. Since $|B| \leqq c$ and since

$$[G \colon A] = [G \colon G_0]\,[G_0 \colon G_1]\,[G_1 \colon G_2]\,[G_2 \colon A] \leqq (a+1)! \cdot c \cdot c \cdot d$$

the result follows.

It is interesting to interpret this result for various fields. If K has characteristic 0 and if B is a finite group, then $K^t[B]$ is semi-simple by Proposition 1.5 of [1]. Thus

COROLLARY 1.5. *Let $K^t[G]$ be a twisted group ring of G over K and let K have characteristic* 0. *Let A be an abelian subgroup of G with $K^t[A]$ commutative.*

(i) *If $[G \colon A] < \infty$ then $K^t[G]$ satisfies a polynomial identity of degree $n = 2[G \colon A]$.*

(ii) *If $K^t[G]$ satisfies a polynomial identity of degree n, then there exists such a group A with $[G \colon A]$ bounded by some fixed function of n.*

COROLLARY 1.6. *Let $K^t[G]$ be a twisted group ring of G over K and let K have characteristic $p > 0$. Let $G \supseteqq A \supseteqq P$ be subgroups of G with P a finite p-group central in A and with $K^t[A]/(JK^t[P] \cdot K^t[A])$ commutative.*

(i) *If $[G \colon A] < \infty$ then $K^t[G]$ satisfies a polynomial identity of degree $n = 2[G \colon A] \cdot |P|$.*

(ii) *If $K^t[G]$ satisfies a polynomial identity of degree n, then there exists suitable A and P with $[G \colon A] \cdot |P|$ bounded by some fixed function of n.*

Proof. Let B be given as in Theorem 1.4 and let P be its normal Sylow p-subgroup. Then P is also central in A. Moreover by Proposition 1.5 of [1] $JK^t[B] = JK^t[P] \cdot K^t[B]$ so the result clearly follows.

Finally in the above if K is a perfect field of characteristic p, then by Lemma 2.1 of [1], $K^t[P] \cong K[P]$ so $K^t[P]/JK^t[P] = K$. It then follows easily that

$$K^t[A]/(JK^t[P] \cdot K^t[A]) \cong K^{t'}[A/P]$$

is in fact some twisted group ring of A/P.

2. **Generalized polynomial identities.** Let E be an algebra over K. A generalized polynomial over E is, roughly speaking, a polynomial in the indeterminates $\zeta_1, \zeta_2, \cdots, \zeta_n$ in which elements of E are allowed to appear both as coefficients and between the indeterminates. We say that E satisfies a generalized polynomial identity if there exists a nonzero generalized polynomial $f(\zeta_1, \zeta_2, \cdots, \zeta_n)$ such that $f(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0$ for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in E$. The problem here is precisely what does it mean for f to be nonzero. For example, suppose that the center of E is bigger than K and let α be a central element not in K. Then E satisfies the identity $f(\zeta_1) = \alpha\zeta_1 - \zeta_1\alpha$ but surely this must be considered trivial. Again, suppose that E is not prime. Then we can choose nonzero $\alpha, \beta \in E$ such that E satisfies the identity $f(\zeta_1) = \alpha\zeta_1\beta$ and this must also be considered trivial. We avoid these difficulties by restricting the allowable form of the polynomials.

We say that f is a multilinear generalized polynomial of degree n if

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} f^\sigma(\zeta_1, \zeta_2, \cdots, \zeta_n)$$

and

$$f^\sigma(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{j=1}^{a_\sigma} \alpha_{0\,\sigma,j}\zeta_{\sigma(1)}\alpha_{1\,\sigma,j}\zeta_{\sigma(2)} \cdots \alpha_{n-1\,\sigma,j}\zeta_{\sigma(n)}\alpha_{n\,\sigma,j}$$

where $\alpha_{i\,\sigma,j} \in E$ and a_σ is some positive integer. This form is of course motivated by Lemma 3.2 of [3]. The above f is said to be nondegenerate if for some $\sigma \in S_n$, f^σ is not a polynomial identity satisfied by E. Otherwise f is degenerate.

In this section we will study group rings $K[G]$ which satisfy nondegenerate multilinear generalized polynomial identities. Let $\Delta = \Delta(G)$ denote the F. C. subgroup of G and let $\theta\colon K[G] \to K[\Delta(G)]$ denote the natural projection.

LEMMA 2.1. *Suppose $K[G]$ satisfies a nondegenerate multilinear generalized polynomial of degree n. Then $K[G]$ satisfies a polynomial identity as given above with*

$$\sum_{j=1}^{a_1} \theta(\alpha_{0,1,j})\theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{n,1,j}) \neq 0 .$$

Proof. Let $K[G]$ satisfy f as above. Since f is nondegenerate, by reordering the ζ's if necessary, we may assume that $f^1(\zeta_1, \zeta_2, \cdots, \zeta_n)$ is not an identity for $K[G]$. Thus since f^1 is multilinear there exists $x_1, x_2, \cdots, x_n \in G$ with

$$\begin{aligned} 0 &\neq f^1(x_1, x_2, \cdots, x_n) \\ &= \sum_{j=1}^{a_1} \alpha_{0,1,j}x_1\alpha_{1,1,j}x_2 \cdots \alpha_{n-1,1,j}x_n\alpha_{n,1,j} . \end{aligned}$$

If we replace ζ_i in f by $x_i\zeta_i$ we see clearly that $K[G]$ satisfies a suitable f with

$$(*) \qquad 0 \neq \sum_{j=1}^{a_1} \alpha_{0,1,j}\alpha_{1,1,j} \cdots \alpha_{n,1,j} .$$

For each i, j write

$$\alpha_{i,1,j} = \sum_k \beta_{ijk}y_k$$

where $\beta_{ijk} \in K[\varDelta]$ and $\{y_k\}$ is a finite set of coset representatives for $\varDelta$ in G. We substitute this into (*) above. It then follows easily that for some $k_0, k_1, \cdots, k_n$ we have

$$0 \neq \sum_{j=1}^{a_1} \beta_{0jk_0}y_{k_0}\beta_{1jk_1}y_{k_1} \cdots \beta_{njk_n}y_{k_n} .$$

Thus if z_i is defined by $z_i = y_{k_0}y_{k_1} \cdots y_{k_{i-1}}$ and $z_0 = 1$ then

$$0 \neq \sum_{j=1}^{a_1} \beta_{0jk_0}^{z_0^{-1}}\beta_{1jk_1}^{z_1^{-1}} \cdots \beta_{njk_n}^{z_n^{-1}} .$$

Now $\beta_{ijk_i} = \theta(\alpha_{i,1,j}y_{k_i}^{-1})$ so

$$\beta_{ijk_i}^{z_i^{-1}} = \theta(z_i\alpha_{i,1,j}y_{k_i}^{-1}z_i^{-1}) = \theta(z_i\alpha_{i,1,j}z_{i+1}^{-1}) .$$

It therefore follows that if we replace ζ_i in f by $z_{i+1}^{-1}\zeta_i z_{i+1}$ and if, in addition, we multiply f on the left by z_0 and on the right by z_{n+1}^{-1}, then this new multilinear generalized polynomial identity obtained has the required property.

LEMMA 2.2. *Let $\alpha_1, \alpha_2, \cdots, \alpha_u, \beta_1, \beta_2, \cdots, \beta_u \in K[G]$. Suppose that for some integers k and t*

$$|\bigcup_i \operatorname{Supp} \alpha_i| = r, \ |\bigcup_i \operatorname{Supp} \beta_i| = s$$

and

$$(\bigcup_i \operatorname{Supp} \alpha_i) \cap \Delta_k(G) \subseteqq \Delta_t(G)$$

with $k \geqq rst^r$. *Let* T *be a subset of* G *and suppose that for all* $x \in G\text{-}T$ *we have*

$$\alpha_1 x \beta_1 + \alpha_2 x \beta_2 + \cdots + \alpha_u x \beta_u = 0 .$$

Then either $[G: T] < (k+2)!$ *or*

$$\theta_k(\alpha_1)\beta_1 + \theta_k(\alpha_2)\beta_2 + \cdots + \theta_k(\alpha_u)\beta_u = 0 .$$

Proof. Let $A = \bigcup_i \operatorname{Supp} \alpha_i$, $B = \bigcup_i \operatorname{Supp} \beta_i$ and write

$$A' = A \cap \Delta_k = \{g_1, g_2, \cdots, g_n\}$$

$$A'' = A - \Delta_k = \{y_1, y_2, \cdots, y_m\}$$

$$B = \{z_1, z_2, \cdots, z_s\} .$$

Here of course $m + n = r$. Set $W = \bigcap_1^n C_G(g_i)$. Since by assumption $A' \subseteqq \Delta_t(G)$ we have clearly $[G: W] \leqq t^r$. Observe that for all $x \in W$, x centralizes $\theta_k(\alpha_i)$.

Suppose that

$$\gamma = \theta_k(\alpha_1)\beta_1 + \theta_k(\alpha_2)\beta_2 + \cdots + \theta_k(\alpha_u)\beta_u \neq 0$$

and let $v \in \operatorname{Supp} \gamma$. If y_i is conjugate to vz_j^{-1} in G for some i, j choose $h_{ij} \in G$ with $h_{ij}^{-1} y_i h_{ij} = vz_j^{-1}$.

Write $\alpha_i = \theta_k(\alpha_i) + \alpha_i''$ and then write

$$\alpha_i'' = \sum a_{ij} y_j, \ \beta_i = \sum b_{ij} z_j .$$

Let $x \in W\text{-}T$. Then we must have

$$\begin{aligned} 0 &= x^{-1}\alpha_1 x \beta_1 + x^{-1}\alpha_2 x \beta_2 + \cdots + x^{-1}\alpha_u x \beta_u \\ &= [\theta_k(\alpha_1)\beta_1 + \theta_k(\alpha_2)\beta_2 + \cdots + \theta_k(\alpha_u)\beta_u] \\ &\quad + [\alpha_1''^x \beta_1 + \alpha_2''^x \beta_2 + \cdots + \alpha_u''^x \beta_u] . \end{aligned}$$

Since v occurs in the support of the first term it must also occur in the second and hence there exists y_i, z_j with $v = y_i^x z_j$ or

$$x^{-1} y_i x = v z_j^{-1} = h_{ij}^{-1} y_i h_{ij} .$$

Thus $x \in C_G(y_i) h_{ij}$. We have therefore shown that

$$W \subseteqq T \cup \bigcup_{ij} C_G(y_i) h_{ij} .$$

Let $w_1, w_2, \cdots, w_d$ be a complete set of coset representatives for W in G. Then $d = [G: W] \leqq t^r$ and the above yields

$$G = Tw_1 \cup Tw_2 \cup \cdots \cup Tw_d \cup S$$

where

$$S = \bigcup_{i,j,c} C_G(y_i)h_{ij}w_c \, .$$

Now the number of cosets in the above union for S is at most

$$rsd \leqq rst^r \leqq k$$

by assumption on k. Moreover $y_i \notin \Delta_k$ so $[G: C_G(y_i)] > k$ for all i. Thus by Lemma 2.3 of [3] $S \neq G$ and then Lemma 2.1 of [3] yields

$$[G: \tilde{T}] \leqq (k+1)!$$

where

$$\tilde{T} = \bigcup\nolimits_c Tw_c \, .$$

Thus

$$[G: T] \leqq (k+1)! \; d \leqq (k+1)! \; (k+2)$$

and the result follows.

We will need the following group theoretic lemma.

LEMMA 2.3. *Let G be a group. The following are equivalent*
(i) $[G: \Delta(G)] < \infty$ *and* $|\Delta(G)'| < \infty$.
(ii) *There exists an integer k with* $[G: \Delta_k(G)] < \infty$.

Proof. Suppose that G satisfies (i) and set $n = [G:\Delta]$, $m = |\Delta'|$. If $x \in \Delta$, then by Theorem 4.4 (i) of [3], $[\Delta: C_\Delta(x)] \leqq m$ and hence $[G: C_G(x)] \leqq nm$. Thus (ii) follows with $k = mn$.

Now suppose that (ii) holds. Since $\Delta(G) \supseteqq \Delta_k(G)$ and $[G: \Delta_k] < \infty$ we conclude that $[G: \Delta] < \infty$. Now $\Delta(G)$ is a subgroup of G so every right translate of Δ_k in G is either entirely contained in Δ or is disjoint from Δ. This implies that $[\Delta: \Delta_k] < \infty$ and say

$$\Delta = \Delta_k y_1 \cup \Delta_k y_2 \cup \cdots \cup \Delta_k y_r \, .$$

Since each $y_i \in \Delta$ we can set $n = \max_i [G: C(y_i)] < \infty$. If $x \in \Delta$ then $x \in \Delta_k y_i$ for some i and this implies easily that $[G: C(x)] \leqq nk$. Thus $[\Delta: C_\Delta(x)] \leqq nk$ and by Theorem 4.4 (ii) of [3], $|\Delta'| < \infty$.

We now come to the main result of this section

THEOREM 2.4. *Let $K[G]$ be a group ring of G over K and sup-*

pose that $K[G]$ *satisfies a nondegegerate multilinear polynomial identity. Then* $[G: \Delta(G)] < \infty$ *and* $|\Delta(G)'| < \infty$.

Proof. By Lemma 2.1. we may assume that $K[G]$ satisfies

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} \sum_{j=1}^{a_\sigma} \alpha_{0,\sigma,j}\zeta_{\sigma(1)}\alpha_{1\,\sigma,j}\zeta_{\sigma(2)} \cdots \alpha_{n-1\,\sigma\,j}\zeta_{\sigma(n)}\alpha_{n,\sigma\,j}$$

with

$$\sum_{j=1}^{a_1} \theta(\alpha_{0,1,j})\,\theta(\alpha_{n,1\,j}) \cdots \theta(\alpha_{n\,1\cdot j}) \neq 0 .$$

We first define a number of numerical parameters associated with f. Set

$$a = \sum_{\sigma \in S_n} \sum_{i=0}^{n} \sum_{j=1}^{a_\sigma} |\operatorname{Supp} \alpha_{i,\sigma\,j}|$$

and

$$r_0 = s_0 = a^{n+1} .$$

Now consider

$$U = \bigcup_{\sigma \in S_n} \bigcup_{j=1}^{a_\sigma} \bigcup_{i=0}^{n} \operatorname{Supp} \theta(\alpha_{i,\sigma,j}) .$$

Then U is a finite subset of $\Delta(G)$ so there exists an integer b with $U \subseteq \Delta_b(G)$. Set

$$t = b^{n+1} \quad \text{and} \quad k = r_0 s_0 t^{r_0} .$$

We assume now that $[G: \Delta_k] = \infty$ and derive a contradiction.

For $i = 0, 1, \cdots, n$ define $S^i \subseteq S_n$ by

$$S^i = \{\sigma \in S_n \mid \sigma(1) = 1,\ \sigma(2) = 2, \cdots, \sigma(i) = i\} .$$

Then $S^0 = S_n$, $S^n = \langle 1 \rangle$ and S^i is just an embedding of S_{n-i} in S_n. We define the multilinear generalized polynomial f_i of degree $n-i$ by

$$\begin{aligned} &f_i(\zeta_{i+1}, \zeta_{i+2}, \cdots, \zeta_n) \\ &= \sum_{\sigma \in S^i} \sum_{j=1}^{a_\sigma} \theta(\alpha_{0\,\sigma,j})\theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-1,\sigma,j})\alpha_{i,\sigma,j}\zeta_{\sigma(i+1)} \cdots \alpha_{n-1,\sigma\,j}\zeta_{\sigma(n)}\alpha_{n\,\sigma\cdot j} . \end{aligned}$$

Thus $f_0 = f$ and

$$f_n = \sum_{j=1}^{a_1} \theta(\alpha_{0,1\cdot j})\theta(\alpha_{1,1\cdot j}) \cdots \theta(\alpha_{n-1,1,j})\alpha_{n\,1,j}$$

is a nonzero element of $K[G]$ since

$$\theta(f_n) = \sum_{j=1}^{a_1} \theta(\alpha_{0,1,j})\theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{n-1,1,j})\theta(\alpha_{n,1,j}) \neq 0 \ .$$

Let $\mathscr{M}$ be the set of monomial polynomials obtained as follows. For each σ, j we start with

$$\alpha_{0,\sigma,j}\zeta_{\sigma(1)}\alpha_{1,\sigma,j}\zeta_{\sigma(2)} \cdots \alpha_{n-1,\sigma,j}\zeta_{\sigma(n)}\alpha_{n,\sigma,j}$$

and we modify it by (1) deleting some but not all of the ζ_i; (2) replacing some of the $\alpha_{i,\sigma,j}$ by $\theta(\alpha_{i,\sigma,j})$; and (3) replacing some of the $\alpha_{i,\sigma,j}$ by 1. Then $\mathscr{M}$ consists of all such monomials obtained for all σ, j and clearly $\mathscr{M}$ is a finite set. Note that $\mathscr{M}$ may contain the zero monomial but it contains no nonzero constant monomial since in (1) we do not allow all the ζ_i to be deleted.

For $i = 0, 1, \cdots, n$ define $\mathscr{M}_i \subseteq \mathscr{M}$ by $\mu \in \mathscr{M}_i$ if and only if $\zeta_1, \zeta_2, \cdots, \zeta_i$ do not occur as variables in μ. Thus $\mathscr{M}_n \subseteq \{0\}$ where 0 is the zero monomial.

Under the assumption that $[G: \Delta_k] = \infty$ we prove by induction on $i = 0, 1, \cdots, n$ that for all $x_{i+1}, x_{i+2}, \cdots, x_n \in G$ either

$$f_i(x_{i+1}, x_{i+2}, \cdots, x_n) = 0$$

or there exists $\mu \in \mathscr{M}_i$ with $\operatorname{Supp} \mu(x_{i+1}, x_{i+2}, \cdots, x_n) \cap \Delta_k \neq \varnothing$. Since $f_0 = f$ is an identity satisfied by $K[G]$ the result for $i = 0$ is clear.

Suppose the inductive result holds for some $i - 1 < n$. Fix $x_{i+1}, x_{i+2}, \cdots, x_n \in G$ and let $x \in G$ play the role of the ith variable. Let $\mu \in \mathscr{M}_i$. If $\operatorname{Supp} \mu(x_{i+1}, \cdots, x_n) \cap \Delta_k \neq \varnothing$ we are done. Thus we may assume that $\operatorname{Supp} \mu(x_{i+1}, \cdots, x_n) \cap \Delta_k = \varnothing$ for all $\mu \in \mathscr{M}_i$. Set $\mathscr{M}_{i-1} - \mathscr{M}_i = \mathscr{N}_{i-1}$.

Now let $\mu \in \mathscr{N}_{i-1}$ so that μ involves the variable ζ_i. Write $\mu = \mu'\zeta_i\mu''$ where μ' and μ'' are monomials in the variables $\zeta_{i+1}, \cdots, \zeta_n$. Then $\operatorname{Supp} \mu(x, x_{i+1}, \cdots, x_n) \cap \Delta_k \neq \varnothing$ implies that

$$x \in h'^{-1}\Delta_k h''^{-1} = \Delta_k h'^{-1}h''^{-1}$$

where $h' \in \operatorname{Supp} \mu'(x_{i+1}, \cdots, x_n)$ and $h'' \in \operatorname{Supp} \mu''(x_{i+1}, \cdots, x_n)$. Thus it follows that for all $x \in G - T$ where

$$T = \bigcup_{\substack{\mu \in N_{i-1} \\ h', h''}} \Delta_k h'^{-1}h''^{-1}$$

we have $\operatorname{Supp} \mu(x, x_{i+1}, \cdots, x_n) \cap \Delta_k = \varnothing$ for all $\mu \in \mathscr{M}_{i-1}$. Thus by the inductive result for $i-1$ we conclude that for all $x \in G-T$ we have $f_{i-1}(x, x_{i+1}, \cdots, x_n) = 0$. Note that T is a finite union of right translates of Δ_k, a subset of G of infinite index.

Now clearly

$$f_{i-1}(x, x_{i+1}, \cdots, x_n)$$
$$= \sum_{\sigma \in S^i} \sum_{j=1}^{a_\sigma} \theta(\alpha_{0\,\sigma,j})\theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-2,\sigma,j})\alpha_{i-1,\sigma,j}x\alpha_{i,\sigma,j}x_{\sigma(i+1)} \cdots \alpha_{n-1\,\sigma,j}x_{\sigma(n)}\alpha_{n\,\sigma\,j}$$
$$+ \sum_{\mu \in \mathscr{M}_i} \mu(x_{i+1}, \cdots, x_n)x\eta(x_{i+1}, \cdots, x_n)$$

where the $\eta(\zeta_{i+1}, \cdots, \zeta_n)$ are suitable monomials. Since

$$f_{i-1}(x, x_{i+1}, \cdots, x_n) = 0$$

for all $x \in G - T$ we can apply Lemma 2.2. However we must first observe that the hypotheses are satisfied.

Let r and s be defined as in Lemma 2.2. Using the basic fact that

$$|\operatorname{Supp} \alpha\beta| \leqq |\operatorname{Supp} \alpha| \; |\operatorname{Supp} \beta|$$

for any $\alpha, \beta \in K[G]$ it follows easily that

$$r \leqq a^{n+1} = r_0, \qquad s \leqq a^{n+1} = s_0 .$$

Now $\mu \in \mathscr{M}_i$ implies that $\operatorname{Supp} \mu(x_{i+1}, \cdots, x_n) \cap \Delta_k = \varnothing$. Therefore the only left hand factors of x which have some support in Δ_k come from the first of the two sums above. Here we have

$$\operatorname{Supp} \theta(\alpha_{i\,\sigma,j}) \subseteqq U \subseteqq \Delta_b$$

and $(\Delta_b)^{n+1} \subseteqq \Delta_{b^{n+1}} = \Delta_t$. Thus the intersection of the supports of these left hand factors with Δ_k is easily seen to be contained in Δ_t. Finally

$$k = r_0 s_0 t^{r_0} \geqq rst^r$$

so the lemma applies.

There are two possible conclusions from Lemma 2.2. The first is that $[G: T] < \infty$. Since T is a finite union of right translates of Δ_k this yields $[G: \Delta_k] < \infty$, a contradiction by our assumption. Thus the second conclusion must hold. Since as we observed above

$$\theta_k(\mu(x_{i+1}, \cdots, x_n)) = 0$$

and clearly

$$\theta_k[\theta(\alpha_{0\,\sigma,j})\theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-2\,\sigma,j})\alpha_{i-1,\sigma,j}]$$
$$= \theta(\alpha_{0,\sigma,j})\theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-2,\sigma,j})\theta(\alpha_{i-1,\sigma,j})$$

we therefore obtain

$$0 = \sum_{\sigma \in S^i} \sum_{j=1}^{a_\sigma} \theta(\alpha_{0\,\sigma\,j})\theta(\alpha_{1\,\sigma\,j}) \cdots \theta(\alpha_{i-1\,\sigma,j})\alpha_{i\,\sigma\,j}x_{\sigma(i+1)} \cdots \alpha_{n-1,\sigma,j}x_{\sigma(n)}\alpha_{n,\sigma,j}$$
$$= f_i(x_{i+1}, x_{i+2}, \cdots, x_n)$$

and the induction step is proved.

In particular, we conclude for $i = n$ that either $f_n = 0$ or there exists $\mu \in \mathscr{M}_n$ with $\operatorname{Supp}\mu \cap \Delta_k \neq \varnothing$. However f_n is known to be a a nonzero constant function and $\mathscr{M}_n \subseteq \{0\}$. Hence we have a contradiction and we must therefore have $[G: \Delta_k] < \infty$. By Lemma 2.3 this yields $[G: \Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$ so the result follows.

3. **Polynomial parts.** Let E be an algebra over K. We say that E has a polynomial part it and only if E has an idempotent e such that eEe satisfies a polynomial identity. In this section we obtain necessary and sufficient conditions for $K[G]$ to have a polynomial part.

We first discuss some well known properties of the standard polynomial s_n of degree n. Here

$$s_n(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} (-1)^\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)} .$$

Suppose A is a subset of $\{\zeta_1, \zeta_2, \cdots, \zeta_n\}$ of size a. Then we let $s_a(A)$ denote s_a evaluated at these variables. This is of course only determined up to a plus or minus sign.

LEMMA 3.1. *Let $a_1, a_2, \cdots, a_r$ be fixed integers with*

$$a_1 + a_2 + \cdots + a_r = n .$$

Then

$$s_n(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{A_1, A_2, \cdots, A_r} \pm s_{a_1}(A_1)s_{a_2}(A_2) \cdots s_{a_r}(A_r)$$

where $A_1, A_2, \cdots, A_r$ run through all subsets of $\{\zeta_1, \zeta_2, \cdots, \zeta_n\}$ with $|A_i| = a_i$ and $A_1 \cup A_2 \cup \cdots \cup A_r = \{\zeta_1, \zeta_2, \cdots, \zeta_n\}$.

Proof. Consider all those terms in the sum for s_n such that the first a_1 variables come from A_1, the next a_2 variables come from A_2, etc. Then the subsum of all such terms is easily seen to be

$$\pm s_{a_1}(A_1)s_{a_2}(A_2) \cdots s_{a_r}(A_r) .$$

This clearly yields the result.

THEOREM 3.2. *Let $K[G]$ be a group ring of G over K which satisfies a polynomial identity. Then $K[G]$ satisfies a standard polynomial identity.*

Proof. If K has characteristic 0 then Theorem 1.1 of [3] and proof of (i) of that theorem show that $K[G]$ satisfies a standard identity. If K has characteristic $p > 0$ then Theorem 1.3 of [3] and a slight modification of the proof of (i) of that theorem show that $K[G]$ satisfies

$$s_{2n}(\zeta_1, \zeta_2, \cdots, \zeta_{2n})s_{2n}(\zeta_{2n+1}, \zeta_{2n+2}, \cdots, \zeta_{4n}) \cdots$$
$$\cdots s_{2n}(\zeta_{2(m-1)n+1}, \zeta_{2(m-1)n+2}, \cdots, \zeta_{2mn}) .$$

Of course it also satisfies this polynomial with all possible permutations of the $2mn$ variables. Thus by Lemma 3.1 $K[G]$ satisfies s_{2mn}.

THEOREM 3.3. *Let $K[G]$ be a group ring of G over K. Then the following are equivalent.*

(i) $[G: \Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$.

(ii) $K[G]$ satisfies a nondegenerate multilinear generalized polynomial identity.

(iii) $K[G]$ has polynomial part.

(iv) $K[G]$ has a central idempotent e such that $eK[G]$ satisfies a standard identity.

Proof. (iv) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (ii). Let e be an idempotent such that $E = eK[G]e$ satisfies a polynomial identity. By Lemma 3.2 of [3], E satisfies an identity of the form

$$g(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} b_\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)} .$$

If $\alpha \in K[G]$ then of course $e\alpha e \in E$. This shows immediately that $K[G]$ satisfies the multilinear generalized polynomial identity

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} b_\sigma e\zeta_{\sigma(1)}e\zeta_{\sigma(2)}e \cdots e\zeta_{\sigma(n)}e .$$

Moreover f is nondegenerate since $b_\sigma \neq 0$ for some σ and then

$$f^\sigma(1, 1, \cdots, 1) = b_\sigma e \neq 0 .$$

(ii) $\Rightarrow$ (i). This follows from Theorem 2.4.

(i) $\Rightarrow$ (iv). Suppose first that K has characteristic 0. Let $H = \Delta(G)'$ so that H is a finite normal subgroup of G. Set

$$e = \frac{1}{|H|} \sum_{x \in H} x \in K[G] .$$

Then e is a central idempotent in $K[G]$ and $eK[G]$ is easily seen to be isomorphic to $K[G/H]$. Now G/H has an abelian subgroup $\Delta(G)/H$ of finite index so by Theorem 3.2 and Theorem 1.1 of [3],

$$eK[G] \cong K[G/H]$$

satisfies a standard identity.

Now let K have characteristic $p > 0$ and let $A = C_{\Delta(G)}(\Delta(G)')$. Then A is normal in G, $[G: A] < \infty$ and $A' \subseteqq \Delta(G)'$ so A' is central in A. Let H be the normal p-compliment of A' and define e as above. Then again e is central in $K[G]$ and $eK[G] \cong K[G/H]$. Since G/H has a p-abelian subgroup A/H of finite index it follows from Theorem 3.2 and Theorem 1.3 of [3] that $K[G/H]$ satisfies a standard identity. This completes the proof of the theorem.

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Received January 28, 1971.

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