PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 3, 1971

UNIQUELY REPRESENTABLE SEMIGROUPS II

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A semigroup S is said to be uniquely representable in terms of two subsets X and Y if $X \cdot Y = Y \cdot X = S$, $x_1y_1, = x_2y_2$ is a nonzero element of S implies $x_1 = x_2$ and $y_1 = y_2$ and $y_1x_1 = y_2x_2$ is a nonzero element of S implies $y_1 = y_2$ and $x_1 = x_2$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

In this paper we are concerned with semigroups S with no zero divisors, $E(S) = \{0, 1\}$, and which are uniquely representable in terms of two subsets X and Y which are iseomorphic copies of the unusual unit interval. Here we show the nonzero elements of the semigroup S can be embedded in a Lie group.

The authors would like to express their appreciation to Professor G. D. Crown for taking part in discussions concerning this paper.

NOTATION. S will represent a semigroup without zero-divisors, $E(S) = \{0, 1\}$ ($E(S)$ is the set of idempotents of S), and which is uniquely representable in terms of X and Y which are isomorphic copies of the usual unit interval. We will let $T = S - \{0\}$ where 0 is the zero of S. Also $X^0 = X - \{0\}$ and $X^{01} = X - \{0, 1\}$ where 1 is the identity for S. Similarly, $Y^0 = Y - \{0\}$, and $Y^{01} = Y - \{0, 1\}$.

Define $\phi: X^0 \times Y^0 \to X^0 \times Y^0$ by $\phi(x, y) = (x', y')$ where x' and y' are the unique elements of X^0 and Y^0 respectively such that $xy = y'x'$. Also define $\psi: X^0 \times Y^0 \to X^0 \times Y^0$ by $\psi(x, y) = (x', y')$ where x' and y' are the unique elements of X^0 and Y^0 respectively such that $yx = x'y'$. It is easy to show ϕ and ψ are homeomorphisms. Also for fixed y, $\pi_1\phi \mid : X^0 \times \{y\} \to X^0$, $\pi_1\psi \mid : X^0 \times \{y\} \to X^0$ are strictly increasing functions, and for fixed x, $\pi_2\phi \mid : \{x\} \times Y^0 \to Y^0$ and

$$\pi_2\psi \mid : \{x\} \times Y^0 \to Y^0$$

are strictly increasing functions.

LEMMA 1. *Let $x_1 \in X^{01}$. If $\pi_2\phi \mid : \{x_1\} \times Y^0 \to Y^0$ is a homeomorphism, then $\pi_2\phi \mid : \{x\} \times Y^0 \to Y^0$ is a homeomorphism for all $x \in X^0$.*

Proof. Fix $\{y_n\}$ a decreasing sequence in Y^0 with $y_n \to 0$, and $x \in X^0$ with $x \geqq x_1$. To show $\pi_2\phi \mid : \{x\} \times Y^0 \to Y^0$ is a homeomorphism we need only show $\pi_2\phi \mid (x, y_n) \to 0$. Let $x_2 \in X^0$ with $x_2x = x_1$. Then there exist sequences $\{q_n\}$, $\{r_n\}$ contained in X^0, $\{s_n\}$, $\{t_n\}$ contained in Y^0 such that $x_1y_n = x_2xy_n = x_2s_nq_n = t_nr_n$. Since

$$t_n = \pi_2\phi \mid (x_1, y_n) \longrightarrow 0, s_n = \pi_2\phi \mid (x, y_n) \longrightarrow 0 .$$

Since $x_1^{2^n} \to 0$, to finish the proof we need only show

$$\pi_2\phi \mid : \{x_1^2\} \times Y^0 \longrightarrow Y^0$$

is a homeomorphism. Select sequence $\{q_n\}, \{r_n\}$ in X^0, $\{s_n\}, \{t_n\}$ in Y^0 such that $x_1^2 y_n = x_1 s_n q_n = t_n r_n$. Since $s_n = \pi_2\phi \mid (x_1, y_n), s_n \to 0$. Thus $t_n = \pi_2\phi \mid (x_1, q_n) \to 0$. Thus $\pi_2\phi \mid : \{x_1^2\} \times Y^0 \to Y^0$ is a homeomorphism. A similar statement for $\pi_1\phi \mid : X^0 \times \{y\} \to X^0$ can be made.

LEMMA 2. *$\pi_2\phi \mid : \{x\} \times Y^0 \to Y^0$ is a homeomorphism for all $x \in X^0$ or $\pi_1\phi \mid : X^0 \times \{y\} \to X^0$ is a homeomorphism for all $y \in Y^0$.*

Proof. Let $x \in X^{01}$ with $\pi_2\phi \mid : \{x\} \times Y^0 \to Y^0$ not a homeomorphism, and let $y \in Y^{01}$. Fix $\{y_n\}$ a decreasing sequence in Y^0 with $y_n \to 0$. There exist sequences $\{q_n\}$ in X^0, $\{s_n\}$ in Y^0 such that

$$xy_n = s_n q_n, \; s_n \not\to 0$$

and $q_n \to 0$. Also there exist sequences $\{r_n\}$ in $X^0, \{t_n\}$ in Y^0 with $q_n y = t_n r_n$. We claim $r_n \to 0$. For if not $t_n \to 0$ and thus

$$xy_n y = s_n t_n r_n \; ,$$

with $s_n t_n \to 0$. However this implies $\pi_2\phi \mid : \{x\} \times Y^0 \to Y^0$ is a homeomorphism. This is a contradiction. So $r_n \to 0$, and thus

$$\pi_1\phi \mid : X^0 \times \{y\} \longrightarrow X^0$$

is a homeomorphism.

LEMMA 3. *T is right reversible or T is left reversible.*

Proof. We will assume $\pi_1\phi \mid : X^0 \times \{y\} \to X^0$ is a homeomorphism for all $y \in Y^0$. We will show T is right reversible. Let $s_1, s_2 \in T$ with $s_1 = x_1 y_1, s_2 = x_2 y_2$ and $y_1 \leqq y_2$. Thus $Ts_1 \cap Ts_2 = Tx_1 y_1 \cap Tx_2 y_2 \neq \phi$ if $Tx_1 y_1 y_2^{-1} \cap Tx_2 \neq \phi$. Let $y_3 x_3 = x_1 y_1 y_2^{-1}$. If $x_3 \leqq x_2$, then

$$Ty_3 x_3 x_2^{-1} \cap T \neq \phi$$

and hence $Ty_3 x_3 \cap Tx_2 \neq \phi$. If $x_2 < x_3$, then $Ty_3 x_3 \cap Tx_2 \neq \phi$ if

$$Ty_3 \cap Tx_2 x_3^{-1} \neq \phi \; .$$

Thus to show T is right reversible we need only show $Tx_4 \cap Ty_4 \neq \phi$ for all $x_4 \in X^{01}, y_4 \in Y^{01}$. Now $\pi_1\phi \mid X^0 \times \{y_4\} \to X^0$ is onto and thus there exists $x_5 \in X^{01}$ such that $\pi_1\phi \mid (x_5, y_4) = x_4$ and thus $x_5 y_4 = y_5 x_4$ for some $y_5 \in Y^0$. Hence $Tx_4 \cap Ty_4 \neq \phi$. If $\pi_2\phi \mid \{x\} \times Y^0 \to Y^0$ is a homeomorphism for all $x \in X^0$, T is left reversible.

Now T is a right (left) reversible cancellative semigroup [2]. Hence

[4] T is algebraically embedded in a group G of left (right) quotents of T. Note that for every element $g \in G$ we have $g = st$ where

$$s, t \in X^0 \cup X^{0^{-1}} \cup Y^0 \cup Y^{0^{-1}} .$$

Also it is easy to see that there exist $x \in X^{01}, y \in Y^{01}$ such that $xT \cap yT \neq \phi$ and $Tx \cap Ty \neq \phi$.

LEMMA 4. *If* $x_1 \in X^{01}$, $y_1 \in Y^{01}$ with $x_1T \cap y_1T \neq \phi$ and $Tx_1 \cap Ty_1 \neq \phi$, *then for* $x_2 \in X^{01}$, $x_2 \geqq x_1$, $y_2 \in Y^{01}$, $y_2 \geqq y_1$ *there exist* $x \in X^{01}$, $y \in Y^{01}$ *such that* $x_2y_2^{-1} = y^{-1}x$.

Proof. Now $Tx_2 \cap Ty_2 \neq \phi$ for $x_2 \geqq x_1$ and $y_2 \geqq y_1$. Thus there exist $s, t \in T$ $sx_2 = ty_2$. Let $s = x_3y_3$ and $t = x_4y_4$. Thus $x_3y_3x_2 = x_4y_4y_2$. If $x_3 < x_4$, then $x_4^{-1}x_3 \in X^{01}$. Thus $x_4^{-1}x_3y_3x_2 = y_4y_2$ or letting $y_3x_2 = x_5y_5$ with $x_5 \in X^0$, $y_5 \in Y^0$ we have $x_4^{-1}x_3x_5y_5 = y_4y_2$. This contradicts S being uniquely representable, so $x_3 \geqq x_4$. Hence $x_3^{-1}x_4 \in X^0$ and thus $y_3x_2 = x_3^{-1}x_4y_4y_2$ or $x_2y_2^{-1} = y_3^{-1}x_3^{-1}x_4y_4$. But $x_3^{-1}x_4 \in X^0$, so there exist $x_6 \in X^0, y_6 \in Y^0$ such that $x_3^{-1}x_4y_4 = y_6x_6$. Hence $x_2y_2^{-1} = y_3^{-1}y_6x_6$. Now $y_3^{-1}y_6 \in Y^{0-}$. For if $y_3^{-1}y_6 \in Y^{01}$ we would have $x_2 = y_3^{-1}y_6x_6y_2$ and letting $x_6y_2 = y_7x_7$ with $x_7 \in X^0$, $y_7 \in Y^0$ we would have $x_2 = y_3^{-1}y_6y_7x_7$ with $y_3^{-1}y_6y_7 \in Y^{01}$, $x_7 \in X^0$. But this contradicts S being uniquely representable. Note that a similar argument yields that there exist $x \in X^0$, $y \in Y^{01}$ such that $y_2x_2^{-1} = x^{-1}y$.

LEMMA 5. *If there exist* $x_1 \in X^{01}$, $y_1, y_2 \in Y^{01}$ *with* $y_1x_1 = x_1y_2$, *then for each* $x \in X^0, y \in Y^0$, *there exist* $y' \in Y^0$ *such that* $yx = xy'$.

Proof. Let $x_1 \in X^{01}$, $y_1, y_2 \in Y^{01}$ with $y_1x_1 = x_1y_2$. We will divide the proof into two parts.

Part 1. We will show that for each $y \in Y^{01}$ there exist $y' \in Y^{01}$ such that $yx_1 = x_1y'$. To prove the above we need only show that there exist $y_3 \in Y^{01}$ such that $\sqrt{y_1}\,x_1 = x_1y_3$. Now $\sqrt{y_1}\,x_1 \in T$ so there exist $x_4 \in X^0$, $y_4 \in Y^0$ such that $\sqrt{y_1}\,x_1 = x_4y_4$. Also let $x_5 \in X^0$, $y_5 \in Y^0$ with $\sqrt{y_1}\,x_4 = x_5y_5$. Now $y_1x_1 = \sqrt{y_1}\sqrt{y_1}\,x_1 = \sqrt{y_1}\,x_4y_4 = x_5y_5y_4$. Thus $x_5 = x_1$ and $y_2 = y_5y_4$. The map $\pi_1\psi \mid : X^0 \times \{\sqrt{y_1}\} \rightarrow X^0$ is strictly increasing and $\pi_1\psi \mid (\pi_1\psi \mid (x_1, \sqrt{y_1}), \sqrt{y_1}) = \pi_1\psi \mid (x_4, \sqrt{y_1}) = x_5 = x_1$, thus $\pi_1\psi \mid (x_1, \sqrt{y_1}) = x_1$. Hence $\sqrt{y_1}\,x_1 = x_1y_4$.

Part 2. To finish the theorem we need only show that there exist $x_2 \in X^{01}$ with $x_2 > x_1$ and $y, y' \in Y^{01}$ such that $yx_2 = x_2y'$. Since the map $s \rightarrow s^2$ is onto we can pick $x_3, x_4 \in X^{01}$, $y_3, y_4 \in Y^{01}$ with $y_1x_1 = (x_3y_3)^2$ and $x_3y_3 = y_4x_4$. Now $y_4x_4x_3y_3 = y_1x_1 = x_1y_2$. Pick $x_5 \in X^{01}$,

$y_5 \in Y^{01}$ such that $x_5y_5 = y_4x_4x_3$. Then $x_1y_2 = y_1x_1 = y_4x_4x_3y_3 = x_5y_5y_3$. Thus $x_5 = x_1$. Select $y_6 \in Y^{01}$ such that $x_1y_5 = y_6x_1$. So

$$y_4x_4x_3 = x_5y_5 = y_5x_1 .$$

Thus $x_4x_3 = x_1$. Hence $x_3 > x_1$. Now there exist $x_0 \in X^{01}$, $y_0 \in Y^{01}$ such that $x_3y_3y_4 = y_0x_0$. So $y_1x_1 = x_3y_3y_4x_4 = y_0x_0x_4$. Hence $x_1 = x_0x_4$. But $x_1 = x_4x_3 = x_3x_4$. Thus $x_0 = x_3$ and $y_0x_3 = y_0x_0 = x_3(y_3y_4)$. This completes the proof.

Let R and R' be the relation $\geqq$ or $\leqq$.

LEMMA 6. *If* $x_1, x_2 \in X^{01}, y_1, y_2 \in Y^{01}$ *with* $x_1y_1 = y_2x_2$, x_1Rx_2, and $y_1R'y_2$, *then for* $x_3, x_4 \in X^{01}, y_3, y_4 \in Y^{01}$ *with* $x_3y_3 = y_4x_4$ *we have* x_3Rx_4 *and* $y_3R'y_4$.

Proof. Consider the map $\pi_1\phi \mid : X^0 \times \{y_1\} \to X^0$, and let

$$\pi_1\phi(x_1, y_1) = x_2$$

and $\pi_1\phi(x_3, y_1) = x_5$. Suppose x_5Rx_3. Now x_1Rx_2 and thus there exist $x \in X^{01}$ such that $\pi_1\phi(x, y_1) = x$. Hence there exist $y \in Y^{01}$ such that $xy_1 = yx$. By Lemma 5 we see $x_3y_1 = y'x_3$ for some $y' \in Y^{01}$. Thus $x_3 = x_5$ and x_3Rx_5. The same type of argument yields $y_1R'y_5$ where $x_3y_1 = y_5x_5$. Applying them again we get x_3Rx_4 and $y_3R'y_4$. This completes the proof.

For $s \in T$, let $s^0 = 1$. Fix $x \in X^{01}$, $y \in Y^{01}$ with $xT \cap yT \neq \phi$ and $Tx \cap Ty \neq \phi$. Now consider G with the topology generated by the following neighborhoods. For t real $t \in (0, 1)$ define

$$N(1, t) = \{x^\alpha y^\beta : \alpha, \beta \in (-t, t)\} .$$

For $g \in G$, $g = sr$ with $s, r \in X^0 \cup X^{0^{-1}} \cup Y^0 \cup Y^{0^{-1}}$. The neighborhoods for g will consist of $sN(1, t)r$ where $N(1, t)$ is a neighborhood of the identity.

LEMMA 7. *If* $N(1, t)$ *is a neighborhood of the identity, then there exist* $N(1, q)$ *a neighborhood of the identity such that*

$$N(1, q) \cdot N(1, q) \subset N(1, t) .$$

Proof. From Lemma 6 and from the fact $y^{1/n} \to 1, x^{1/n} \to 1$ we can pick N such that for $n > N$ the following hold: (1) $y^{1/n}x^q = x_ny_n$ and $x_n \in (x^{t/4}, 1]$ implies $x^q \in (x^{t/2}, 1]$, (2) $x^{1/n} \in (x^{t/4}, 1]$, and (3)

$$y^{1/n}x^q = x'_ny'_n$$

with $x^q \in (x^{t/2}, 1]$ implies $y'_n \in (y^{t/2}, 1]$.

From Lemma 4 there exist $\bar{x}_n \in X^{01}, \bar{y}_n \in Y^{01}$ such that

$$y^{-1/n}x^{1/n} = \bar{x}_n\bar{y}_n^{-1} .$$

Since $x^{1/n} = y^{1/n}y^{-1/n}x^{1/n} = y^{1/n}\bar{x}_n\bar{y}_n^{-1}$ we see that $y^{1/n}\bar{x}_n = x^{1/n}\bar{y}_n$. Thus from the above $\bar{x}_n \in (x^{t/2}, 1]$ and $\bar{y}_n \in (y^{t/2}, {}^1]$ or $\bar{y}_n^{-1} \in [1, y^{-t/2})$. That is there exist N such that for

$$n > N, \psi(x^{1/n}, y^{-1/n}) \subset \{(x^\alpha, y^\beta)\colon \alpha, \beta \in (-t/2, 0) \cup (0, t/2)\} .$$

Using the same procedure we can find M large enough such that

$$\{\psi(x^{1/M}, y^{1/M}), \psi(x^{-1/M}, y^{1/M}), \psi(x^{1/M}, y^{-1/M}), \psi(x^{-1/M}, y^{-1/M}), (x^{1/M}, y^{1/M}) , \\ (x^{-1/M}, y^{-1/M})\} \subset \{x^\alpha, y^\beta)\colon \alpha, \beta \in (-t/2, 0) \cup (0, t/2)\} .$$

Now by Lemma 4 and Lemma 6

$$\{y^\alpha x^\beta\colon \alpha, \beta \in (-1/M, 1/M)\} \subset \{x^\alpha y^\beta\colon \alpha, \beta \in (-t/2, t/2)\} .$$

Hence $N(1, 1/M)\ N(1, 1/M) \subset N(1, t)$.

LEMMA 8. *G is a topological semigroup.*

Proof. To prove this we need only show that for each

$$s \in X^0 \cup X^{0^{-1}} \cup Y^0 \cup Y^{0^{-1}}$$

and $N(1, t)$ a neighborhood of the identity there exist $N(1, q)$ a neighborhood of the identity such that $sN(1, q) \subset N(1, t)s$. We will assume $s \in Y^0 \cup Y^{0^{-1}}$. Now $N(1, t)s = \{x^\alpha s y^\beta\colon \alpha, \beta \in (-t, t)\}$. Now pick r such that $\{sx^\alpha\colon \alpha \in (-r, r)\} \subset \{x^\alpha y^\beta s\colon \alpha, \beta \in (-t/2, t/2)\}$. Set $q = \min\{r, t/2\}$. Then $sN(1, q) \subset N(1, t)s$.

Now G is a locally compact topological semigroup which is algebraically a group. By [9] G is a topological group. Moreover since G is locally euclidean [8] G is a two-dimensional Lie group.

THEOREM 9. *T is embedded in G.*

Proof. The inclusion map $i\colon T \to G$ is an iseomorphism into.

It should be pointed out here that an alternate and more general method for embedding semigroups in groups has been constructed by D. R. Brown and Michael Friedberg [4].

COROLLARY 10. *If D is a uniquely divisible semigroup on the two-cell with $E(D) = \{0, 1\}$ ($E(D)$ is the set of idempotents for D), then $D - \{0\}$ is embedded in a Lie group.*

Proof. In [2] it was shown that $D - \{0\}$ is uniquely representable in terms of two usual unit intervals. Thus $D - \{0\}$ is embedded in a Lie group.

Examples and characterization. The authors would like to extend their appreciation to J. Lawson for supplying us with the information for the characterization of the uniquely representable semigroups.

(1) Let $(I, \cdot)$ denote the closed unit interval with the usual multiplication. Then $(I, \cdot) \times (I, \cdot)/[(\{0\} \times I) \cup (I \times \{0\})]$ is the only commutative which is uniquely representable in terms of two usual unit intervals [6], [7].

If S is non-abelian, then G is a non-abelian Lie group and G can be represented by the real matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0$ [1].

In the examples below we will take S to be the semigroup induced by one point compactification of the subsemigroups of G. The point added will always be the zero for S.

It is to be noted that Example 4 is anti-isomorphic to Example 2 and Example 5 is anti-isomorphic to Example 3.

(2) Let S be the topological semigroup generated by taking the one point compactification of the semigroup of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0, y \geqq 0, x + y \leqq 1$. Note S is uniquely divisible and thus S is uniquely representable in terms of two usual unit intervals [2]. Also S is not left reversible. It is easy to see that if W is the semigroup induced by the one point compactification of any collection of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $0 < x \leqq 1$ and $y \geqq \alpha(x - 1)$, $y \leqq \beta(x - 1)$ for two real numbers α and β, W is iseomorphic to S.

(3) The one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $0 < x \leqq 1, y \geqq 0$ is a uniquely divisible semigroup on the two-cell. S is uniquely representable in terms of $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \cup \{0\}$ and $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cup \{0\}$. This semigroup is both left and right reversible. Furthermore,

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & xy \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Also if W is the one point compactification of any semigroup of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $y \geqq \alpha(x - 1), 0 < x \leqq 1$ or $y \leqq \alpha(x - 1), 0 < x \leqq 1$ for some real number α, then S is iseomorphic to W. We will say S is half commutative if for each $x \in X^0, y \in Y^0$ there exists $y' \in Y^0$ such that $xy = y'x$.

(4) Let S be the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x \geqq 1, y \geqq 0, y \leqq x - 1$. Then S is uniquely divisible,

left but not right reversible, it is not half commutative. Also if W is the semigroup formed by the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x \geqq 1, y \geqq \alpha(x-1), y \leqq \beta(x-1), \beta > \alpha$, W is iseomorphic to S.

(5) Consider the semigroup S formed by the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ $x \geqq 1, y \geqq 0$. S is uniquely divisible, half commutative, right and left reversible. S differs from Example 3, since S has no copy of Example 2 contained in it, but Example 3 has a copy of Example 2 in it. Also if W is the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ $x \geqq 1, y \geqq \alpha(x-1)$, α real, or $x \geqq 1, y \leqq \alpha(x-1)$, then W is iseomorphic to S.

These are all of the semigroups which are uniquely representable in terms of two usual unit intervals. Note that they are all uniquely divisible.

COROLLARY 11. *If S is uniquely representable in terms of two usual unit intervals and without zero divisor and $E(S) = \{0, 1\}$, then S is uniquely divisible.*

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Received May 4, 1970 and in revised form October 16, 1970,

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