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Γ-EXTENSIONS OF IMAGINARY QUADRATIC FIELDS

ROBERT GOLD

Let p be an odd rational prime and $E_0 = \mathbb{Q}(\sqrt{-m})$ a quadratic imaginary number field. There is a unique Γ-extension E of E_0 for the prime p which is absolutely abelian. For each positive integer n there is a subfield E_n of E which is cyclic of degree p^n over E_0 and by Iwasawa the exponent of p in the class number of E_n is of the form $\mu p^n + \lambda n + c$ for sufficiently large n. We here examine the analytic formula for the class number of E_n and in the case $p = 3$ give a simple condition implying that $\mu = 0$. It follows easily from this condition that there are infinitely many imaginary quadratic fields which have Γ-extensions for the prime 3 with the invariants $\mu = 0$ while $\lambda \geqq 1$.

1. **Analytic formula.** Let $\mathbb{Q}$ be the rationals, p an odd prime, n an integer $\geqq 0$, and $\zeta_{p^{n+1}}$ a primitive p^{n+1} root of unity. Let F_n be the subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ of degree p^n over the rationals so that $F_n/\mathbb{Q}$ is cyclic, p is the unique ramified prime for the extension, and p is totally ramified. Let $E_0 = \mathbb{Q}(\sqrt{-m})$, a quadratic imaginary field where $(m, p) = 1$ and let $E_n = F_n.E_0$, the composite field.

We attempt to study the order, e_n, to which p divides the class number of E_n,

$$h_{E_n} = p^{e_n}\cdot h' \qquad (p, h') = 1$$

by use of the classical analytic formula for an arbitrary number field k:

$$\lim_{s\to 1}(s-1)\zeta_k(s) = \frac{2^{s+t}\pi^t R_k}{m_k\sqrt{|D_k|}}h_k \tag{1}$$

where, as usual, R_k is the regulator of k; m_k, the order of the group of roots of unity; D_k, the discriminant of k; and s and t, the number of real and complex infinite primes of k.

We note the following sequence of lemmas:

LEMMA 1. *$m_{E_n} = m_{F_n} = 2$ unless $E_0 = \mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-1})$.*

Proof. By degrees: $[E_n : \mathbb{Q}] = 2p^n$.

Note that in the two excluded cases $(p, m_{E_n}) = 1$ if $(p, m) = 1$.

LEMMV 2. $D_{E_n} = D_{F_n}^2 \cdot D_{E_0}^{p^n}$ *and* $D_{F_n} = p^{t_n}$; $t_n = (n+1)p^n - (p^n - 1)/(p-1) - 1$.

Proof. First statement is trivial, second is proved as follows.

Note that $\zeta_{p^{n+1}}$ is a distinguished element for the extension $\mathcal{Q}(\zeta_{p^{n+1}})/F_n$ in the relation its different bears to the different of the extension [3]. The computation of the different of $\mathcal{Q}(\zeta_{p^{n+1}})/F_n$ becomes then an exercise in determinants. The result combined with the well known different of $\mathcal{Q}(\zeta_{p^{n+1}})/\mathcal{Q}$ gives the expression above.

LEMMA 3. $R_{E_n} = R_{F_n} \cdot 2^a$ *some* $a \in Z$.

Proof. F_n is the maximal real subfield of E_n and the result is then well known [1].

Now let $k = E_n$, respectively F_n, in equation (1) and divide the former by the latter. Taking into account the preceding lemmas this simplifies to:

$$\lim_{s \to 1} (\zeta_{F_n}(s)/\zeta_{F_n}(s)) = \frac{2^a \pi^{p^n}}{\sqrt{|D_{E_0}|^{p^n}}} \frac{h_{E_n}}{p^{s_n} h_{F_n}} \tag{2}$$

$$s_n = \frac{1}{2} t_n = \frac{1}{2}((n+1)p^n - (p^n - 1)/(p-1) - 1) .$$

On the other hand $\zeta_{E_n}(s) = \prod L(s, \chi)$ where the product is taken over all Dirichlet characters belonging to the extension $E_n/\mathcal{Q}$. Since $g(E_n/\mathcal{Q}) \cong \mathcal{Z}/2 + \mathcal{Z}/p^n$ we can write $\zeta_{E_n}(s) = \prod L(s, \chi_0^i \chi_1^j)$, $i = 0, 1$; $j = 0, \cdots, p^n - 1$ where χ_0, χ_0^2 are the characters belonging to $E_0/\mathcal{Q}$ while $\chi_1^0, \cdots, \chi_1^{p^n - 1}$ are the characters belonging to $F_n/\mathcal{Q}$. Hence $\zeta_{F_n}(s) = \prod L(s, \chi_1^j)$, $j = 0, \cdots, p^n - 1$ and therefore $\zeta_{E_n}(s)/\zeta_{F_n}(s) = \prod L(s, \chi_0 \chi_1^j)$, $j = 0, \cdots, p^n - 1$. Furthermore the χ_1^{pk}, $k = 0, \cdots, p^{n-1} - 1$ are the characters belonging to $F_{n-1}/\mathcal{Q}$ and therefore

$$\frac{\zeta_{E_n}(s)/\zeta_{F_n}(s)}{\zeta_{E_{n-1}}(s)/\zeta_{F_{n-1}}(s)} = \prod_{\substack{0 < j < p^n \\ (j,p)=1}} L(s, \chi_0 \chi_1^j) . \tag{3}$$

Note in passing that χ_1 is an even character and takes on the p^nth roots of unity as values. Comparing (2) and (3) we may write

$$\prod_{\substack{0 < j < p^n \\ (j,p)=1}} L(1, \chi_0 \chi_1^j) = \frac{h_{E_n} \cdot h_{F_{n-1}} \pi^{\varphi(p^n)}}{h_{F_n} h_{E_{n-1}} p^{(s_n - s_{n-1})} \sqrt{|D_{E_0}|^{\varphi(p^n)}}} . \tag{4}$$

Note that χ_0 is primitive modulo $d = D_{E_0}$ = the conductor of $E_0/\mathcal{Q}$, while χ_1^j, $(j, p) = 1$ is primitive modulo p^{n+1} = the conductor of $F_n/\mathcal{Q}$. It follows that $\chi_0 \chi_1^j$, $(j, p) = 1$ is primitive with modulus $w = dp^{n+1}$ and is an odd character. It is well known then that

$$L(1, \chi_0\chi_1^j) = \frac{\pi i \tau(\chi_0\chi_1^j)}{w^2} \sum_{\substack{0<k<w \\ (k,w)=1}} \chi_0\bar{\chi}_1^j(k)k \tag{5}$$

where $\tau(\chi_0\chi_1^j)$ is the classical Gauss sum and $|\tau(\chi_0\chi_1^j)| = \sqrt{w}$. Comparing now (4) and (5) and taking absolute values we see

$$\frac{\left|\prod_{\substack{(j,p)=1 \\ 0<j<p^n}} \sum_{\substack{(k,w)=1 \\ 0<k<w}} \chi_0\bar{\chi}_1^j(k)k\right|}{d^{\varphi(p^n)} p^{(n+1)\varphi(p^n)}} = \frac{h_{E_n} h_{F_{n-1}}}{h_{F_n} h_{E_{n-1}}} . \tag{6}$$

Next we examine the sum appearing in (6).

$$\begin{aligned} S_j &= \sum_{0<k<w} \chi_0\bar{\chi}_1^j(k)k = \sum_{\alpha=0}^{d-1} \sum_{i=0}^{p^{n+1}-1} \chi_0\bar{\chi}_1^j(i + \alpha p^{n+1})(i + \alpha p^{n+1}) \\ &= \sum_{\alpha=0}^{d-1} \sum_{i=0}^{p^{n+1}-1} \chi_0(i + \alpha p^{n+1})\bar{\chi}_1^j(i)i + \alpha p^{n+1} \sum_{i=0}^{p^{n+1}} \bar{\chi}_1^j(i)\chi_0(i + \alpha p^{n+1}) . \end{aligned}$$

But since

$$\sum_{\alpha=0}^{d-1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i)\chi_0(i + \alpha p^{n+1})i = \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i)i \sum_{\alpha=0}^{d-1} \chi_0(i + \alpha p^{n+1}) = 0$$

we have

$$S_j = p^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \sum_{\alpha=0}^{d-1} \alpha\chi_0(i + \alpha p^{n+1}) .$$

We now make the following assumption for the sake of simplifying notation and proofs: (A) $p^{n+1} \equiv 1(d)$. It then follows that

$$S_j = p^{n+1} \sum_i \bar{\chi}_1^j(i) \sum_\alpha \chi_0(i\alpha + \alpha) .$$

Letting $w_k = \sum_{\alpha=0}^{d-1} \alpha\chi_0(\alpha + k)$ one can easily deduce that $w_0 = w_1$, $w_{k+d} = w_k$, and $w_k = w_0 + d\sum_{\alpha=0}^{k-1} \chi_0(\alpha)$. Then

$$\begin{aligned} S_j &= p^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i)w_0 + d \sum_{\alpha=0}^{i-1} \chi_0(\alpha) \\ &= p^{n+1}w_0 \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) + d \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \sum_{\alpha=0}^{i-1} \chi_0(\alpha) \\ &= dp^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i)\cdot\alpha_i ; \quad \text{where} \quad \alpha_i = \sum_{\alpha=0}^{i-1} \chi_0(\alpha) . \end{aligned}$$

Comparing this last result with (6) we see that

$$\prod_{\substack{(j,p)=1 \\ 0<j<p^{n+1}}} \sum_{i=0}^{p^{n+1}-1} \alpha_i\bar{\chi}_1^j(i) = \frac{h_{E_n} h_{F_{n-1}}}{h_{F_n} h_{E_{n-1}}} , \tag{7}$$

and again $\alpha_i = \sum_{\alpha=0}^{i-1} \chi_0(\alpha)$.

We reduce our concern now to the power of p occurring in each member of (7). By results of Iwasawa $(p, h_{F_n}) = (p, h_{F_{n-1}}) = 1$ while for sufficiently large n: $\mathrm{ord}_p(h_{E_n}) = \mu p^n + \lambda n + c$, $\mathrm{ord}_p(h_{E_{n-1}}) = \mu p^{n-1} + \lambda(n-1) + c$ ([2]). Therefore

$$\mathrm{ord}_p \prod_{0<j<p^{n+1}} \sum_{i=0}^{p^{n+1}-1} \alpha_i \bar{\chi}_1^j(i) = \mu\varphi(p^n) + \lambda \,. \tag{8}$$

It is clear that $\alpha_i \in \mathscr{Z}$ and hence $\sum_{i=0}^{p^{n+1}-1} \alpha_i \bar{\chi}_1^j(i)$ is an integer in $\mathscr{Q}(\zeta_{p^n})$. In fact, $\prod \sum \alpha_i \bar{\chi}_1^j(i)$ is simply the absolute norm of this integer. Hence

$$\begin{aligned} \mu\varphi(p^n) + \lambda &= \mathrm{ord}_p \mathscr{N}_Q\left(\sum_{i=0}^{p^{n+1}-1} \alpha_i \chi_1(i)\right) \\ &= \mathrm{ord}_{\mathfrak{p}} \sum_{i=0}^{p^{n+1}-1} \alpha_i \chi_1(i) \,. \end{aligned} \tag{9}$$

Here $\mathfrak{p}$ is the unique prime of $\mathscr{Q}(\zeta_{p^n})$ dividing p.

We now rewrite $\sum \alpha_i \chi_1(i)$ in terms of an integral basis of $\mathscr{Q}(\zeta_{p^n})$. Let g be a primitive root modulo p^{n+1}, i.e. $\bar{g}$ generates $(\mathscr{Z}/p^{n+1})^*$. Then $\sum_{i=0}^{p^{n+1}-1} \alpha_i \chi_1(i) = \sum_{s=0}^{\varphi(p^{n+1})-1} \alpha_{g_s} \chi_1(g^s)$ where $0 < g_s < p^{n+1}$ and $g_s \equiv g^s(p^{n+1})$. Then $\eta = \chi_1(g)$ is a primitive p^nth root of unity and

$$\sum_{s=0}^{\varphi(p^{n+1})-1} \chi_1(g^s)\alpha_{g_s} = \sum_{s=0}^{\varphi(p^{n+1})-1} \eta^s \alpha_{g_s} \,.$$

Since $1, \eta, \cdots, \eta^{\varphi(p^n)-1}$ form a $\mathscr{Z}$-basis for the integers of $\mathscr{Q}(\zeta_{p^n})$ we may rewrite this last sum, using identities of the form $1 + \eta^{p^{n-1}} + \cdots + \eta^{(p-1)p^{n-1}} = 0$, as

$$\sum_{s=0}^{\varphi(p^{n+1})-1} \eta^s \alpha_{g_s} = \sum_{s=0}^{\varphi(p^n)-1} \eta^s \sum_{i=0}^{p-2} (\alpha_{g_{s+ip^n}} - \alpha_{g_{\varphi(p^n)+t+ip^n}})$$

where $0<t<p^{n-1}$ and $t \equiv s(p^{n-1})$. It follows from (9) then that

$$\mu\varphi(p^n) + \lambda = \mathrm{ord}_{\mathfrak{p}} \sum_{s=0}^{\varphi(p^n)-1} \eta^s \sum_{i=0}^{p-2} (\alpha_{g_{s+ip^n}} - \alpha_{g_{\varphi(p^n)+t+ip^n}}) \,. \tag{10}$$

For sufficiently large n the left member of (10) is $\geqq \varphi(p^n)$ if and only if $\mu > 0$. However the right member is greater than $\varphi(p^n)$ if and only if $\mathfrak{p}^{\varphi(p^n)} = (p)$ divides the algebraic integer in brackets. Since this integer is written in terms of an integral basis it is divisible by (p) if and only if the coefficients of η^s is divisible by p for every s. Hence $\mu > 0$ if and only if p divides

$$\sum_{i=0}^{p-2} (\alpha_{g_{s+ip^n}} - \alpha_{g_{\varphi(p^n)+t+ip^n}}) \qquad s = 0, 1, \cdots, \varphi(p^n) - 1 \,. \tag{11}$$

2. **Special case of $p = 3$.** If we specialize to $p = 3$, $s = 0$ we

may proceed in the following manner. For $p=3$, $s=0$ equation (11) reads

$$\alpha_{g_0}+\alpha_{g_{3^n}}-\alpha_{g(3^n)}-\alpha_{g_{3^n+\varphi(3^n)}}\,. \tag{12}$$

Clearly $g_0=1$, $g_{3^n}=3^{n+1}-1$; while for appropriate choice of g we have $g_{\varphi(3^n)}=3^n+1$ (resp. 2.3^n+1) and $g_{\varphi(3^n)+3^n}=2\cdot 3^n-1$ (resp. 3^n-1). Hence (12) reads, letting $M(m)=\sum_{\alpha=0}^{m}\chi_0(\alpha)$,

$$\begin{aligned}&M(0)+M(3^{n+1})-M(3^n)-M(2\cdot 3^n-2)\\&(\text{resp. } M(0)+M(3^{n+1}-2)-M(2\cdot 3^n)-M(3^n-2))\,.\end{aligned} \tag{13}$$

Clearly $M(0)=0$ and recalling that (A) $3^{n+1}\equiv 1$ (d) we see that $M(3^{n+1}-2)=M(d-1)=0$ as well. Since $\chi_0(-1)=-1$ we have the trivial but useful identity $M(m)=M(kd-m-1)$, $kd-m-1>0$. By this it follows that $M(2\cdot 3^n-2)=M(kd+1-3^n-2)=M(kd-3^n-1)=M(3^n)$ (resp. $M(3^n-2)=M(2\cdot 3^n)$). Hence (13) reduces to $-2M(3^n)$ (resp. $-2M(2\cdot 3^n)$) and so $\mu>0$ if and only if $M(3^n)\equiv 0$ (3) (resp. $M(2\cdot 3^n)\equiv 0$ (3)).

Again by (A): $M(2\cdot 3^n)=M(kd+1-3^n)=M(3^n-2)=M(3^n)-\chi_0(3^n)-\chi_0(3^n-1)$. Since both congruences above must be satisfied it follows that $\mu>0$ if and only if $\chi_0(3^n)+\chi_0(3^n-1)\equiv 0$ (3). Multiplying by $\chi_0(3)\neq 0$ we have $[\chi_0(3^n)+\chi_0(3^n-1)]=\chi_0(3)=\chi_0(1)-\chi_0(2)$. Hence we may finally state in the language of Iwasawa

THEOREM. *Let $E_\infty=\bigcup E_n$ be the absolutely abelian Γ-extension for the prime* 3 *of $\mathcal{Q}(\sqrt{-m})$; $(m,3)=1$. If* 2 *does not split in $\mathcal{Q}(\sqrt{-m})/\mathcal{Q}$ then the invariant μ equals* 0.

EXAMPLE 1. $E_0=\mathcal{Q}(\sqrt{-5})$. Since $\chi_0(3)=+1$, 3 splits in $\mathcal{Q}(\sqrt{-5})/\mathcal{Q}$ and it is easy to see from the structure of the genus field for E_n/E_0 that $\lambda\geqq 1$. On the other hand, $\chi_0(2)=0$ and therefore $\mu=0$. Obviously all $\mathcal{Q}(\sqrt{-m})$ for $m\equiv 7,10$ (12) behave in this manner.

EXAMPLE 2. $E_0=\mathcal{Q}(\sqrt{-23})$. This field has class number 3 and is therefore of some interest. Unfortunately $\chi_0(2)=1$, but we may use the remark above that $\mu>0$ if and only if $M(3^n)\equiv 0$ (3). By (A): $M(3^n)=M(3^{-1})=M(8)$ in this case. But $M(8)=4\not\equiv 0$ (3) and so again $\mu=0$.

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OHIO STATE UNIVERSITY