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APPROXIMATION AND INTERPOLATION

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Let X be a compact plane set, X^0 its interior, and suppose E is a subset of $\partial X = X \backslash X^0$. $H^\infty(X^0)$ is the algebra of all bounded analytic functions on X^0 and $H_E^\infty(X^0)$ denotes all bounded continuous functions on $X^0 \cup E$ analytic in X^0.

Interpolation sets for $H_E^\infty(X^0)$ are studied if E is open relative to ∂X.

If X satisfies certain conditions which involve analytic capacity, it is shown that an interpolation set S for $H^\infty(X^0)$ is an interpolation set for $H^\infty(0)$ for some open set 0 which contains every point of X except the points on ∂X in the closure of S. Similar results are proved for $R(X)$ without restrictions on X.

Introduction and notation. The paper is divided into three sections. Section 1 is intended as a motivation for the problems to be studied in the next section. We here prove a simple approximation result for $H_B^\infty(X^0)$ in case $X = \{z: |z| \leqq 1\}$ and apply it to interpolation problems.

The proofs here are based on the theory of H^p-spaces, but the ideas behind the proofs are the same as in the next sections.

In §2 we use the techniques developed by A. G. Vitushkin to generalize the results of section 1 and the main theorem of Heard and Wells in [6]. We also here make use of some results from [5].

In §3 we prove the results for $H^\infty(X^0)$ and $R(X)$ concerning interpolation and analytic continuation.

Problems of this kind were first studied by Akutowicz and Carleson in [1]. Later one of their results has been extended but only in case $X = \{z: |z| \leqq 1\}$ (See [3], [6] and [8]).

We have defined $H_B^\infty(X^0)$ above. If $B = \partial X$ we define as usual $H_B^\infty(X^0) = A(X)$. We say that $A(X)$ is pointwise boundedly dense in $H^\infty(X^0)$ if every bounded analytic function on X^0 is a pointwise limit of a bounded sequence from $A(X)$.

Whenever S is a topological space $C(S)$ denotes all bounded complex valued continuous functions f on S and we put $||f|| = \sup\{|f(x)|, x \in S\}$. If $S \subset \boldsymbol{C}$ we always assume it has the topology induced from $\boldsymbol{C}$.

If f is a complex valued function defined on a set S and F is a subset of S, we define $||f||_F = \sup\{|f(x)|: x \in F\}$.

The basic results from the theory of analytic capacity and rational approximation will be used several times. A convenient reference is [4, ch. VIII]. (See also [9] or [10]).

If E is a subset of the complex plane, $\gamma(E)$ and $\alpha(E)$ denote the analytic capacity and the continuous analytic capacity of E respectively.

If $\delta > 0$, $z \in C$ then $\Delta(z, \delta)$ denotes the open disc with radius δ centered at z. If f is a bounded measurable function on C we put $\|f\| = \|f\|_\infty$ where $\| \ \|_\infty$ is the essential supremum of $|f|$ with respect to plane Lebesgue measure.

S^2 denotes the extended complex plane with the usual topology.

1. Let $D = \{z: |z| < 1\}$ and $T = \partial D$ be the circle group. If u is a real integrable function on T we define the analytic function $H_u(z)$ by

$$H_u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(\theta) d\theta \qquad z \in D .$$

It is well known (see [7] p. 67) that we can factor an $f \in H^\infty(D)$, into $f = f_1 f_2$ where $f_1, f_2 \in H^\infty(D)$, $\lim_{r\to 1} |f_1(re^{i\theta})| = 1$ a.e., on T and $f_2 = \lambda \exp(H_u)$ where $|\lambda| = 1$ and H_u is as above. f_1 is called an inner function, f_2 an outer function.

A Blaschke product is an inner function given by a product

$$B(z) = \lambda \cdot z^k \cdot \prod_n \frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \text{ where } \sum_n (1 - |\alpha_n|) < \infty .$$

The convergence is uniform on compact subsets of C at a positive distance from the set $\{1/\bar{\alpha}_n | n = 1, 2, \cdots\}$. Let $E \subset T$ be compact and $B = T \backslash E$. Using the notation above we now have:

THEOREM 1.1. *Suppose $F \subset D$ is closed relative to $D \cup B$. Given $h \in H^\infty(D)$ and $\varepsilon > 0$ there exists $f \in H_B^\infty$ such that $\|h - f\|_F < \varepsilon$ and $\|f\| \leqq \|h\|$. If h is an inner (outer) function we can choose f to be inner (outer).*

Theorem 1 follows from the factorization Theorem mentioned above and the following three lemmas:

LEMMA 1.1. *Every inner function $f \in H_B^\infty$ is a uniform limit of Blaschke products in H_B^∞.*

LEMMA 1.2. *Given $\varepsilon > 0$ and a Blaschke product $G \in H^\infty(D)$ there exists a Blaschke product $G' \in H_B^\infty$ such that $\|G - G'\|_F < \varepsilon$.*

LEMMA 1.3. *Given $u \leqq 0$ in $L^1(T)$ and $\varepsilon > 0$ there exists an outer function $G \in H_B^\infty$ such that $\|G - \exp H_{(u)}\|_F < \varepsilon$ and $\|G\| \leqq 1$.*

Lemma 1.1. is proved in [7, p. 176].

Proof of Lemma 1.2. Let $B = \bigcup_1^\infty J_n$ where the union is disjoint and each J_n is a half-open arc such that every compact subset K of

B is covered by a finite number of the arcs. Let $D_n = \{z \in D \backslash \{0\}: z/|z| \in J_n\}$. Assume dist $(J_n, F) > 0$ for each n. From G we take a subproduct G_1 having only a finite number of factors with zeros in D_1 and such that

$$||G - G_1||_F < \frac{\varepsilon}{2^2}.$$

Then we take a subproduct G_2 of G_1 having only a finite number of factors with zeros in D_2 and such that $||G_1 - G_2||_F < \varepsilon/2^3$. We proceed in this way and get a sequence of Blaschke-products G_n.

Let G' be the subproduct of G which contains precisely the factors common to all G_n.

Since $G_n \to G'$ uniformly on compact subsets of the domain of definition of G' we clearly have $||G - G'||_F < \varepsilon$.

Proof of Lemma 1.3. Let $u_1 = u|_B$ and $u_2 = u - u_1$. Choose a realvalued function $v \in L'(T)$ continuously differentiable on B such that $v \leqq 0$, $v = 0$ on E and such that $\sup_{z \in F} |H_{(u_1)}(z) - H_{(v)}(z)| < \varepsilon/e$. Then the function $G = \exp [H_{(v+u_2)}]$ is the required one. If u_1 has compact support v is easy to find. In the general case u_1 can be written as a sum of such functions and then an "$\varepsilon/2^n$-argument" works.

COROLLARY 1.1. *If $H^\infty|_F$ is equal to $C(F)$ then $H_B^\infty|_F = H^\infty|_F$.*

Corollary 1.1 is an immediate consequence of Theorem 1.1 and the following lemma that will be useful to us several times:

LEMMA 1.4. *Suppose $T: X \to Y$ is a linear continuous map from a Banach space X into a Banach space Y and there exist numbers $t \in \langle 0, 1 \rangle$ and $M < \infty$ such that for every $y \in Y$ with $||y|| \leqq 1$ there exist $x \in X$ such that $||y - Tx|| < t$ and $||x|| \leqq M$.*

Then $TX = Y$ and if $y \in Y$ then $y = Tx$ for some $x \in X$ with $||x|| \leqq M/(1 - t)$.

Proof. Let $y \in Y$ and $||y|| = 1$. Choose a sequence $\{x_n\} \subset X$ such that $||x_n|| \leqq Mt^n$, $n = 1, 2, \cdots$ and such that $||T(\sum_1^m x_n) - y|| \leqq t^m$ for $m = 1, 2, \cdots$. Then $x = \sum_1^\infty x_n \in X$, $||x|| \leqq M/(1 - t)$ and $Tx = y$. The corollary follows now by letting $X = H_B^\infty$, $Y = H^\infty|F \cap D$ and $T: X \to Y$ be the restriction map.

2. We now generalize Theorem 1.1.

THEOREM 2.1. *Suppose $A(X)$ is pointwise boundedly dense in $H^\infty(X^0)$. Then there exists a constant k such that if $B \subset \partial X$ is open* *relative to* ∂X, $h \in H^\infty(X^0)$, $F \subset X^0$ *is closed relative to* $X^0 \cup B$ *and* $\varepsilon > 0$ *we can find a function* $f \in H_B^\infty(X)$ *such that* $\|f\| \leqq k\|h\|$ *and* $\|f - h\|_F < \varepsilon$.

Proof. From the hypothesis we get constants c and r such that

$$\gamma(\Delta(z, \delta)\backslash X^0) \leqq c\alpha(\Delta(z, r\delta)\backslash X^0) \tag{1}$$

whenever $z \in \boldsymbol{C}$, $\delta > 0$.

That (1) in fact is equivalent with the hypothesis of the theorem follows from Theorem 2.2 in [5].

Choose for a fixed $\delta > 0$ and $k = 1, 2, \cdots$, points $z_{k\delta}$ and functions $\phi_{k\delta}$: $\Delta(z_{k\delta}, \delta) \to [0, 1]$ such that

(i) $\phi_{k\delta} \in C_0^1 \Delta(z_{k\delta}, \delta)$

(ii) $\sum_1^\infty \phi_{k\delta} \equiv 1$ in $\boldsymbol{C}$

(iii) $\|\partial\phi_{k\delta}/\partial\bar{z}\| \leqq 4/\delta$

(iv) No complex number is contained in more than 25 of the discs $\Delta_k = \Delta(z_{k\delta}, \delta)$ (See more about this construction in Ch. VIII in [4]).

If f is a bounded measurable function on $\boldsymbol{C}$ and $\phi \in C_0(\boldsymbol{C})$ we define

$$T_\phi(f)(\zeta) = \frac{1}{\pi}\iint \frac{f(z) - f(\zeta)}{z - \zeta}\frac{\partial\phi}{\partial\bar{z}}(z)\, dxdy\,. \tag{2}$$

We have by Stokes theorem:

$$T_\phi(f)(\zeta) = f(\zeta)\cdot\phi(\zeta) + \frac{1}{\pi}\iint \frac{f(z)}{z - \zeta}\frac{\partial\phi}{\partial\bar{z}}(z)\, dxdy\,. \tag{3}$$

By (2) $T_\phi(f)$ is analytic wherever f is analytic and by (3) $T_\phi(f)$ is continuous whenever f is continuous.

We also have (4) $\|T_\phi(f)\|_\infty \leqq 4$ diam $Y\cdot\partial\phi/\partial\bar{z}\|_\infty\ \|f|_X\|_\infty$ where Y is the support of ϕ. Let us also remark that $f - T_\phi(f)$ is analytic in the interior of the set $\{z: \phi(z) = 1\}$ and that

$$T_\phi(f)'(\infty) = \frac{-1}{\pi}\iint f(z)\frac{\partial\phi}{\partial\bar{z}}(z)\, dxdy$$

which follows from (3) above.

We now prove Theorem 2.1. If $h \in H^\infty(X^0)$ we assume it extended to a measurable function on $\boldsymbol{C}$ bounded by $\|h\|_{X^0}$.

For $n = 1, 2, \cdots$ we choose open sets V_n and compact sets $K_n \subset V_n$ such that $V_n \cap V_m = \varnothing$ if $|n - m| > 1$ and $B = \bigcup_1^\infty K_n$ and $V_n \cap F = \varnothing$ for every n. We also require that $K \cap \bar{V}_n \neq \varnothing$ only for finitely many n if $K \subset \boldsymbol{C}\backslash((\partial X)\backslash B)$ is compact.

For each n we choose $\delta_n > 0$ and functions $\{\phi_{kn}\}_{k=1}^\infty$ supported on discs Δ_{kn} with radius δ_n as described above.

We put $G_{kn} = T_{\phi_{kn}}(h)$ and $I_n = \{k: \bar{\Delta}_{kn} \cap K_n \neq \emptyset\}$. The inequality (1): $\gamma((z, \delta)\backslash X^0) \leqq C\alpha(\Delta(z, r\delta)\backslash X^0)$ for $z \in C$, $\delta > 0$ combined with the proof of "the principal lemma" (See p. 174–176 in [9] or p. 211–213 in [4]) gives the existence of functions $H_{kn} \in C(S^2)$ analytic outside a compact subset of $\Delta(z_{kn}, (r + 2)\delta_n)\backslash X^0$ such that $G_{kn} - H_{kn}$ has a triple zero in the series development about ∞ and $||H_{kn}|| \leqq a||h||$ where a depends only on c and r.

Exactly as in [5] at p. 192–193 we get that $f_n = \sum_{I_n}(G_{kn} - H_{kn})$ satisfies

(a) $||f_n|| \leqq b||h||$

(b) If $\{w: |z - w| \leqq N\delta_n\} \cap \Delta_{kn} = \emptyset$ for any $k \in I_n$ then $|f_n(z)| \leqq b||h|| \sum_{N+1}^{\infty} 1/m^2$.

Here b depends on c and r. Suppose now $1 > \varepsilon > 0$. By (b) we have if $(\delta_n)^{-1}$ dist $(K_n, C\backslash V_n)$ is sufficiently large:

(c) $$||f_n||_{C/V_n} < \frac{\varepsilon}{2^{n+1}} ||h|| .$$

Put $g = h - \sum_{n=1}^{\infty} f_{2n-1}$. Then $||g|| \leqq ||h||(1 + \varepsilon + b) < (b + 2)||h||$, g is analytic in X^0 and continuous on K_{2n-1} for $n = 1, 2, \cdots$. The continuity follows since

$$g = (h - f_{2n-1}) + (\sum_{j \neq n} f_{2j-1})$$

and

$$h - f_{2n-1} = h - T_\phi h + \sum_{I_{2n-1}} H_{i\,j}$$

where $\phi = \sum_{I_{2n-1}} \phi_i$ equals one in a neighbourhood of K_{2n-1}. We now replace h by g, and (K_{2n-1}, V_{2n-1}) by (K_{2n}, V_{2n}), for $n = 1, 2, \cdots$ and repeat the argument. Since the T_ϕ-operator preserves analyticity and continuity we get a function $f \in H_B^\infty(X^0)$ satisfying $||f|| \leqq (b + 2)^2||h||$ and $||h - f||_F \leqq ||h - g||_F + ||g - f||_F < \varepsilon$ since $V_n \cap F = \emptyset$ for every n.

A subset F of $X^0 \cup B$ is called a peak set for $H_B^\infty(X^0)$ if whenever $0 \supset F$ is open relative to $X^0 \cup B$ and $\varepsilon > 0$, there exists $f \in H_B^\infty(X^0)$ such that $f = 1 = ||f||$ on F and $|f| < \varepsilon$ on $(X \cup B)\backslash 0$. Peak interpolation set is defined in the same way.

THEOREM 2.2. *Let X, B be as above and assume the hypothesis of theorem* 2.1. *Then we have for a set S closed relative to $X^0 \cup B$:*

(i) *If each compact subset of $S \cap B$ is a peak interpolation set for $A(X)$ then $S \cap B$ is a peak interpolation set for $H_B^\infty(X^0)$. If in addition $L = H^\infty(X^0)|_{S\backslash B}$ is closed in $C(S\backslash B)$ then $H_B^\infty|S$ consists of all bounded* *continuous functions f on S such that $f|_{S\setminus B} \in L$*

(ii) *conversely if S is a (peak) interpolationset for H_B^∞ then every compact subset of S is a (peak) interpolationset for $A(X)$ and $H^\infty(X^0)|_{S\setminus B} = C(S\setminus B)$.*

To prove Theorem 2.2 we need to generalize Lemma 3 of [6]. The next lemma is stated for the algebra H_B^∞ but the result is valid in the setting of a general sup norm algebra defined on a compact Hausdorff space. (With $H_B^\infty(X^0)$ replaced by an algebra of functions defined in a natural way).

LEMMA 2.1. *Suppose $K \subset B$ is closed relative to B and every compact subset of K is a peak interpolation set for $A(X)$.*

Then for every $g \in C(K)$ we can find $f \in H_B^\infty(X^0)$ such that $f|K = g$ and $|f| < ||g||$ on $X^0 \cup B\setminus K$.

Proof. Put $Y = X^0 \cup B$ and choose open sets V_n and compact sets $K_n \subset V_n$ such that $K = \bigcup_1^\infty K_n$ and $V_n \cap V_m = \emptyset$ if $|n - m| > 1$. We also require $\bar{V}_n \cap M \neq \emptyset$ only for finitely many n if $M \subset Y$ is compact. Put $V_0 = K_{-1} = K_0 = \emptyset$ and $f_0 \equiv 0$. Choose $g \in C(K)$ with $||g|| = 1$ and let $t \in \langle 0, 1 \rangle$.

We shall construct a sequence $\{f_n\} \subset A(X)$ such that we have for $k = 0, 1, 2, \cdots$.

(1, k): $f_k = g - f_{k-1}$ on K_k

(2, k): $||f_k|| \leqq 1 + t$

(3, k): $|f_k| < t2^{-k-1}$ on $(X\setminus V_k) \cup K_{k-1}$

(4, k): $f_k = 0$ on $(K_{k-1} \cap K_k) \cup (K_{k+1} \cap K_{k+2})$

(5, k): $||f_k - g||_{K_{k+1}} < 1 + t$.

Assume $f_0, \cdots, f_n$ constructed and $n \geqq 0$.

Choose now $h \in A(X)$ satisfying $(i, n + 1)$ for $i = 1, 2$. Then $h = 0$ on $K_n \cap K_{n+1}$ (this is trivial if $n = 0$ and follows from $(4, n - 1)$ if $n \geqq 1$ since then $h = g - (g - f_{n-1}) = f_{n-1}$ on $K_n \cap K_{n+1}$.

Since K_{n+1} is a peak-set it is now clear how to modify h so also $(3, n + 1)$ is valid. By $(4, n)$ $h = g$ on $K_{n+1} \cap K_{n+2}$.

Therefore $|h - g| < t$ on a subset W of K_{n+2} open in the relative topology. We assume $W \cap V_{n+3} = \emptyset$. Choose now $h_1 \in A(X)$ such that $h_1 = 1 = ||h_1||$ on K_{n+1}, $h_1 = 0$ on $K_{n+2}\setminus W$ and $0 \leqq h_1 \leqq 1$ on $K_{n+1} \cup K_{n+2}$. We can assume $W \supset K_{n+1} \cap K_{n+2}$.

If we put $f_{n+1} = hh_1$ then we have what we want.

Since the initial step $(n = 0)$ is trivial, we now have constructed a sequence $\{f_n\}$ such that $F_n = \sum_1^n f_k$ satisfies

(1) $||F_n||_X \leqq 2(1 + t) + t < 5$

(2) F_n converges uniformly on compact subsets of Y and has a limit F in H_B^∞ with $\|F\| \leqq 5$.

Suppose now that $x \in K$. Then $x \in K_n$ for some n. By $(1, n)$ $f_{n-1}(x) + f_n(x) - g(x) = 0$ and by $(3, k)$ for $k = 1, 2, \cdots$ $|F(x) - g(x)| < \sum_{n=1}^{\infty}(t \cdot 2^{-n-1}) = t$. By Lemma 1.4 every g is equal to $f|K$ for some $f \in H_B^\infty$ with $\|f\| \leqq 5/(1-t) \leqq 6$ if t is small.

Having established this partial result we look at the proof and see that it shows the following:

LEMMA 2.2. *Given $\varepsilon > 0$ and subset $F \subset Y$ closed in Y for which $F \cap K = \varnothing$.*

Then there exists a function f in H_B^∞ such that $f \equiv 1$ on K, $|f| < \varepsilon$ on F and $\|f\| < 6$.

Proof. Assume in the proof above that $g \equiv 1$ on K and $V_n \cap F = \varnothing$ for all n and choose the functions small on F.

From Lemma 2.2 and the fact that if $g \in C(K)$ $g = f|K$ with $f \in H_B^\infty$ and $\|f\| \leqq 6\|g\|$ we can prove Lemma 2.1. In fact the rest of the proof follows from Lemma 4.4, Lemma 4.5 and Theorem 4.6 in [2]. In [2] only functions on a compact set Ω are studied. But if Ω is replaced by Y and compact subsets of Ω are replaced by closed subsets of Y, the results in [2] can be used word for word.

We now prove the rest of (i) in Theorem 2.2:

Let $\varepsilon > 0$ and put $M = \{h \in C(S): h|_{S \subset X} \in H^\infty(X^0)|S \cap X^0\}$.

It is sufficient to prove that $H_B^\infty|_S = M$. Clearly $H_B^\infty|_S \subset M$. Assume $h \in M$ and $\|h\| = 1$. Choose by Lemma 2.1 $f_1 \in H_B^\infty$ such that $f_1 = h$ on $S \cap B$ and $\|f_1\| \leqq 1$.

Since $H^\infty(X^0)|S \cap X^0$ is closed in $C(S \cap X^0)$ there exists by the open mapping theorem a constant k_1 independent of $h - f_1$ and $f_2^1 \in H^\infty(X^0)$ with $\|f_2^1\| \leqq k_1\|h - f_1\| \leqq 2k_1$ such that $f_2^1 = h - f_1$ on $S \cap X^0$. Choose by Theorem 2.1 a function $f_2 \in H^\infty_{(\partial X)\setminus\bar{S}}(X^0)$ with $\|f_2\| \leqq k\|f_2^1\|$ and $\|f_2 - f_2^1\|_{S \cap X^0} < \varepsilon$.

Choose an open set $V \supset S \cap B$ such that $\max(|f_2|, |f_2^1| < 2\varepsilon$ on $V \cap S \cap X^0$ and by Lemma 2.1 $f_3 \in H_B^\infty(X^0)$ such that $f_3 \equiv 0$ on $S \cap B$, $\|f_3\| \leqq 2$ and $|1 - f_3| < \varepsilon$ on $S \setminus V$.

Put $f = f_1 + f_2 f_3$. Then $\|f\| \leqq 1 + 2kk_1$ and $\|h - f\|_S < 6\varepsilon$. Choosing $\varepsilon < 1/6$ we have that $H_B^\infty(X^0)|_S = M$ by Lemma 1.4.

Proof of (ii). That $H^\infty(X^0)|_{S\setminus B} = C(S\setminus B)$ is a simple normal family argument.

To prove the statements about $A(X)$ one takes a compact subset K of S and $g \in C(K)$ with $\|g\| \leqq 1$. There exists a constant k independent of g and $g_1 \in H_B^\infty$ such that $g_1 = g$ on K and $\|g_1\| \leqq k$. Fix

$t \in \langle 0, 1 \rangle$.

As in the proof of Theorem 2.1 we fix $\delta > 0$ and choose functions G_n and H_n corresponding to g_1 and δ and put $f_\delta = g_1 - \sum_I (G_n - H_n)$ where $I = \{n: \Delta(z_n, \delta) \cap ((\partial X)\backslash B) \neq \emptyset\}$.

Let V be open and containining $(\partial X)\backslash B$ and assume $\bar{V} \cap K = \emptyset$. We have $||f_\delta|| \leqq kk_1$ and where k_1 depends only on c and r appearing in (1) in the proof of Theorem 2.1.

Choosing δ small compared with dist (K, V) we get

(a) $||f_\delta - g||_K < t$ and

(b) $||g_1 - f_\delta|| \leqq k_1 ||g_1||_V$.

From (a) and Lemma 1.4 one get that $A(X)|_K = C(K)$ since $f_\delta|_X \in A(X)$.

Suppose now S is a peak interpolation set for H_B^∞. Let F be a compact subset of X disjoint from K and $\varepsilon > 0$. Let $\varepsilon' > 0$ and choose g_1 such that $g_1 = g$ on $K ||g_1|| = ||g||$ and $|g_1| < \varepsilon'$ on $F \cup V$. Choosing ε' small and remembering (b) we can get f_δ as small as we please on F. Then as in the proof of Lemma 1.4 we can find $h \in A(X)$ such that $h|_K = g$, $|h| < \varepsilon$ on F and $||h|| \leqq 1/(1 - t)$. But then K is a peak interpolation set for $A(X)$. (This follows as in the proof of Lemma 2.1).

3. We assume in this section $\emptyset \neq U = X^0$ for some compact subset X of $\boldsymbol{C}$.

THEOREM 3.1. *Suppose S is a relatively closed subset of U and $H^\infty(U)|_S$ is a closed subspace of $C(S)$. Suppose there exist constants c and r such that* (*): $\gamma(\Delta(z, \delta)\backslash U) \leqq c\gamma(\Delta(z, r\delta)\backslash X)$ *whenever* $z \in \boldsymbol{C}$, $\delta > 0$ *and* $\Delta(z, \delta) \cap \bar{S} = \emptyset$. *Then there exists an open set* $0 \supset X\backslash(\bar{S}\backslash S)$ *such that* $H^\infty(0)|_S = H^\infty(U)|_S$.

COROLLARY 3.1. *Suppose $A(X)$ is p.b. dense in $H^\infty(U)$ and every $f \in A(X)$ belongs locally to $R(X)$ in $X\backslash(\bar{S}\backslash S)$. Then the conclusion of Theorem 3.1. holds.*

Proof of the corollary. The hypothesis of the corollary implies via Vitushkin's theorem (Thm. 8.1. on p. 214 in [4]) and Theorem 2.2 of [5] that (*) holds.

The proof of Theorem 3.1 starts with the following lemma:

LEMMA 3.1. *Assume the hypothesis of Theorem 3.1.*

Suppose $K \subset (\partial X)\backslash(\bar{S}\backslash S)$ is compact and $V \supset K$ is open. Let $\varepsilon > 0$.

There exists an open set $V_0 \supset K$ and a constant M such that if $h \in L^\infty(dxdy)$, $h|_U \in H^\infty(U)$ and $||h||_\infty = 1$ we can find a bounded function f on $\boldsymbol{C}$ analytic in $X^0 \cup V_0$ such that $||f - h||_{C\backslash V} < \varepsilon$ and $||f - h|| <$

$M\|h\|_V$ *where* M *depends only on* c *and* r.

Proof. From the hypothesis we have $(*)$ $\gamma(\Delta(z, \delta)\backslash U) \leqq c\gamma(\Delta(z, r\delta)\backslash X)$ if $z \in \boldsymbol{C}$, $\delta > 0$ and $\Delta(z, \delta) \cap \bar{S} = \phi$. Suppose ϕ is continuously differentiable and supported on $\Delta = \Delta(z, \delta)$. Then by $(*)$:

$$\text{(I)} \qquad |T_\phi h'(\infty)| = \left|\frac{1}{\pi}\iint h \frac{\partial \phi}{\partial \bar{z}}\, dxdy\right| \leqq 4\delta c\|h\|_\Delta \left\|\frac{\partial \phi}{\partial \bar{z}}\right\| \gamma(\Delta(z, r\delta)\backslash X) .$$

Now we use some of the notation from §2. Put $G_k = T_{\phi_{k\delta}}(h)$ where $\phi_{k\delta} \in C_0^1(\Delta(z_k, \delta))$ and $E_k = \Delta(z_k, (r+2)\delta)\backslash X$. Then by (I) we have if $\Delta_k = \Delta(z_k, \delta)$

$$\text{(II)} \qquad |G_k'(\infty)| \leqq 16c\|h\|_{\Delta_k}\gamma(E_k) .$$

Let W_k be the analytic center of E_k and $\beta(E_k)$ the analytic diameter of E_k. (See [4] on p. 209 for definitions).

Let $I = \{k\colon \Delta_k \cap K \neq \varnothing\}$. We can assume $V \cap \bar{S} = \varnothing$ and δ chosen so small that $\Delta(z_{k'}(r+2)\delta) \subset V$ if $k \in I$. Then it follows from the proof of (iii) $\Rightarrow$ (i) in Theorem 8.1 in [4] that

$$\text{(III)} \quad \left|\beta(G_k, W_k)\right| = \left|\frac{1}{\pi}\iint h(z - W_k)\frac{\partial \phi_k}{\partial \bar{z}}\right| \leqq c\cdot k(r)\|h\|_V\, \gamma(E_k)\beta(E_k)$$

where $k(r)$ depends only on r.

Now it follows from Lemma 6.3 on p. 209 in [4] that there exist functions $f_1^{(k)}$, $f_2^{(k)}$ analytic outside a compact subset of E_k such that $\|f_1^{(k)}\| + \|f_2^{(k)}\| \leqq 20$ and such that $0 = f_1^{(k)}(\infty) = f_2^{(k)}(\infty) = (f_1^{(k)})'(\infty) = \beta(f_2, W_k)$, $(f_2^{(k)})'(\infty) = \gamma(E_k)$ and $\beta(f_1, W_k) = \gamma(E_k)\cdot\beta(E_k)$.

But then we can choose complex numbers a and b such that $H_k = a\, f_1^k + b\, f_2^k$ satisfies:

(i) $G_k - H_k$ has a triple zero at infinity

(ii) $\|H_k\| \leqq$ c $M(r)\|h\|_V$ where $M(r)$ depends only on r.

It is important that the singularities of H_k depends only on the singularities of f_1^k and f_2^k.

Define now $f = h - \sum_I (G_k - H_k)$. We have $\|G_k - H_k\| \leqq a\|h\|_V$ where a depends only on r and c.

Since dist $(K, \boldsymbol{C}\backslash V) > 0$ we can exactly as at p. 193 in [5] show that $\|f - h\|_{C\backslash V} < \varepsilon$ if δ is small. It is important that we can use the same δ for any h satisfying $\|h\|_\infty \leqq 1$.

Since $\sum_I \phi_k \equiv 1$ in a neighbourhood of K the function $h - \sum_I G_k = h - T_{\Sigma_I \phi_k}(h)$ is analytic in U and in a neighbourhood V' of K. (V' depends only on δ). Remembering how the functions H_k were chosen we have proved the lemma.

The proof of the next lemma is almost a copy of an argument from §2.

LEMMA 3.2. *Suppose* $\varepsilon > 0$ *and assume the hypothesis of Theorem* 3.1. *There exists a constant* k *and an open set* $0 \supset X \backslash (\bar{S} \backslash S)$ *such that if* $h \in H^\infty(U)$ *and* $\|h\| \leqq 1$, *there exists* $f \in H^\infty(O)$ *such that* $\|f\| < k$ *and* $\|f - h\|_S < \varepsilon$.

Proof. We put $\partial X \backslash (\bar{S} \backslash S) = \bigcup_1^\infty K_n$ where each K_n is compact and $V_n \supset K_n$ is open such that $V_n \cap V_m \neq \emptyset \Rightarrow |n - m| \leqq 1$, and $V_n \cap \bar{S} = \emptyset$ for all n and $V_n \cap K \neq \emptyset$ only for finitely many n if $K \subset C \backslash (\bar{S} \backslash S)$ is compact.

Looking at the functions H_k constructed in Lemma 3.1 and noting that if f is a bounded measurable function $T_{\phi_k}(f)$ is analytic wherever f is analytic, we see that the technique used in the proof of Theorem 2.1 combined with Lemma 3.1 yields a function f such that

(i) $\|f\|_\infty \leqq 2M + 2 \overset{\text{def}}{=} k$ (M is as in Lemma 3.1)

(ii) $\|f - h\|_S < \varepsilon$

(iii) f is analytic in U and in an open set containing $\bigcup_1^\infty K_n$ and this open set does not depend on h.

We now prove Theorem 3.1. Since $H^\infty(U)|_S$ is closed in $C(S)$ there exists a constant L such that every $g \in C(S)$ equals $h|_S$ where $h \in H^\infty(U)$ *and* $\|h\| \leqq L\|g\|$. Let $\varepsilon > 0$.

We choose the open set 0 as in Lemma 3.2 and apply the lemma to h/L. We get a function $f_1 \in H^\infty(0)$ such that $\|f_1\| \leqq k$ (k is as in Lemma 3.2).

Then the function $f = Lf_1$ satisfies:

(i) $\|g - f\|_S \leqq \varepsilon L$

(ii) $\|f\| \leqq kL$.

If we choose $\varepsilon < 1/L$ we can prove Theorem 3.1 via Lemma 1.4.

There is a result similar to Theorem 3.1 *for the the function* $R(X)$:

THEOREM 3.2. *Let* X *be a compact subset of* $\mathbf{C}$ *and* $E \subset X$ *a closed subset. If* $R(X)|_E$ *is closed in* $C(E)$ *then there exists a compact set* $Y \supset X$ *such that* $X \backslash E \subset Y^0$ *and* $R(Y)|_E = R(X)|_E$.

If E *is a peak set for* $R(X)$ *then it also is for* $R(Y)$.

Proof. We proceed in exactly the same way as in the proof of Theorem 3.1.

Instead of the condition (*) to get the functions $f_1^{(k)}$ and $f_2^{(k)}$ used in Lemma 3.1 we now use Theorem 8.1 on p. 214 in [4].

$$\text{Let } A = \{h \in C(S^2) \colon h|_X \in R(X)\}\,.$$

Consider the following map T defined on A: Given $h \in A$ then modify it to $f = h - \sum_1^\infty f_n$ exactly as in the proof of Theorem 3.1. Then (as in the same proof) modify f to

$$T(h) = f - \sum_n g_n = h - \sum_n f_n - \sum_n g_n .$$

We can arrange it so that

(i) $\|f_n\| \leqq c_1 \sup\{|h(z_1) - h(z_2)|: z_1, z_2 \in \boldsymbol{C}, |z_1 - z_2| \leqq 1/n\}$

(ii) $\|f_n\|_{C\setminus W_n} \leqq c_1\|h\|\cdot 2^{-n}$

where c_1 is an absolute constant. These inequalities also hold if f_n is replaced by g_n and h is replaced by f.

It follows that T maps A into $C(S^2)$. Since the T_ϕ operator is linear it follows that T also is and we have $\|T(h)\| \leqq a\|h\|$ where a is an absolute constant.

From the proof of Theorem 3.1 it follows that there exists an open set $0 \supset X\setminus E$ such that $T(h)$ is analytic in 0 for all h in A. Choose a locally finite covering of $(\partial X)\setminus E$ consisting of closed discs Δ_i such that $\Delta_i \subset 0$ for each i.

Put $Y = XU(\bigcup_1^\infty \Delta_i)$.

To prove that $Th|_Y \in R(Y)$ for $h \in A$ we may assume h is analytic in a neighbourhood of X. If h is such a function then $f_n \equiv 0$ for all but a finite number and the same holds for $\{g_n\}$. This is because $T_\phi(h) = 0$ whenever $\phi \in C^1$ and h is analytic in a neighbourhood of the support of ϕ. But then it is easy to see that $Th|_Y \in R(Y)$.

It remains to show that E is a peak for $R(Y)$ if it is a peak set for $R(X)$. Suppose therefore that $F \subset Y$ is compact and $F \cap E = \varnothing$.

Let $g \in R(X)|_E$. Let $\varepsilon_1, \varepsilon_2 > 0$. Choose $h \in A$ such that $h = g$ on E, $\|h\| = \|g\|$ and $|h| < \varepsilon_1$ on $\boldsymbol{C}\setminus W$ where W is an open set containing E to be specified. Assume $\|g\| \leqq 1$.

We shall have $F \subset \boldsymbol{C}\setminus W$, Recall that $Th = (h - \sum_1^\infty f_n) - \sum_1^\infty g_n$.

Let N be a number such that $|\sum_m^\infty f_n(z)| \leqq \varepsilon_2$ and $|\sum_m^\infty g_n(z)| \leqq \varepsilon_2$ whenever $m \geqq N$, $\|h\| \leqq 1$ and $z \notin \bigcup_{2m-1}^\infty V_n$ and $h \in A$.

We assume $\boldsymbol{C}\setminus W \supset V_n$ if $n < 2N + 1$ and $F \cap V_n = \varnothing$ if $n \geqq N$.

From the way f_n is constructed we have $\|f_n\| \leqq c_2\|h\|_{V_{2n-1}}$ and $\|f_n\|_{C\setminus W_{2n-1}} \leqq c_2\|h\|_{V_{2n-1}}\cdot 2^{-n}$ where c_2 is an absolute constant.

These inequalities also hold if f_n is replaced by g_n and h is replaced by $h - \sum_1^\infty f_n$ and V_{2n-1} is replaced by V_{2n}.

But then $\|h - \sum_1^{N-1} f_n - \sum_1^{N-1} g_n\|_F \leqq \varepsilon_1 + \max\{\|f_n\|, n = 1 \cdots N - 1\} + \max\{\|g_n\|: n = 1, \cdots N - 1\} < \varepsilon_1 + c_2\varepsilon_1 + c_2((c_2 + 1)\varepsilon_1 + \varepsilon_2)$.

But if now $1/2 > \varepsilon_3 > 0$ is given, $g \in R(X)|_E$ and $\|g\| \leqq 1$ we can choose ε_1, ε_2 and h such that

$$\|g - Th\|_E < \varepsilon_3 < 1/2 \tag{1}$$

$$\|Th\|_F < \varepsilon_3 \tag{2}$$

$$\|Th\| \leqq c_3\|h\| \leqq c_3 \tag{3}$$

where c_3 is absolute.

But then by Lemma 1.4 there exists $f \in R(Y)$ such that

$$f|_E = g, \|f\| \leqq \frac{c_3}{1 - 1/2} = 2c_3 \quad \text{and} \quad \|f\|_F < 2c_3\varepsilon_3 .$$

Since c_3 is absolute and F and ε_3 are arbitrary E is a peak-set for $R(Y)$. The rest of the proof is essentially Bishop's "1/4 − 3/4 theorem". (See Thm. 2.1 on p. 5 in [10]).

Let $HR(X)$ denote the set of all functions on X^0 which are pointwise limits on X^0 of bounded sequences in $R(X)$.

LEMMA 3.3. *A bounded analytic function f on X^0 is in $HR(X)$ if and only if there exist constants c and r such that*

$$(*) \; |(T_\phi \tilde{f})'(\infty)| \leqq c\delta \left\| \frac{\partial \phi}{\partial \bar{z}} \right\| \gamma(\Delta(z, r\delta)\backslash X) \|\tilde{f}\|$$

for some bounded measurable extension $\tilde{f}$ of f to C and whenever $z \in C$, $\delta > 0$ and $\phi \in C_0^1(\Delta(z, \delta))$.

Proof. If $(*)$ holds for some bounded measurable extension $\tilde{f}$ of f then $f \in HR(X)$ by the proof of Thm. 8.1 at p. 214 in [4].

Using this theorem and assuming $\{f_n\} \subset R(X)$, $\|f_n\| \leqq k\|f\|$ and $f_n \to f$ pointwise on X^0 one get that $(*)$ is valid for some extension $\tilde{f}$ of f being a w^* clusterpoint of $\{f_n\}$ in $L^\infty(dxdy)$. (We assume each f_n extended to $C(S^2)$ and still bounded by $\|f_n\|_X$).

Using Lemma 3.3 one can prove the following result:

THEOREM 3.3. *If X is compact, $S \subset X^0$ is closed relative to X^0 and $HR(X)|_S$ is closed in $C(S)$ then $HR(Y)|_S = HR(X)|_S$ for some $Y \supset X$ such that $Y^0 \supset X\backslash(\bar{S}\backslash S)$.*

Proof. Define for each f *in* $HR(X)$ $\|f\| = \inf\{\sup_n \|f_n\|: f_n \in R(X), f_n \to f$ pointwise on $X^0\}$. Then $HR(X)$ is a Banach space.

Applying the open mapping theorem to the restriction map $HR(X) \to C(S)$, Theorem 3.3 is proved in the same way as Theorem 3.1. The set Y is constructed as in Theorem 3.2.

Comments on Theorem 3.1. Suppose A is subspace of $H^\infty(U)$ and $T_\phi \bar{h}|_U \in A$ whenever $\bar{h}$ is a bounded measurable extension of an $h \in A$ and ϕ is continuously differentiable with compact support. We shall then say that A is invariant under T_ϕ.

The following result holds:

Suppose $A \subset H^\infty(U)$ is invariant under T_ϕ and $h|_U \in A$ whenever

h is analytic in a neighbourhood of X. If it is possible to choose the sets $\{V_n\}$ appearing in the proof of Theorem 3.1 in such a way that $f \in A$ whenever $f = \lim_n f_n$ where $\{f_n\}$ is a bounded sequence from A and the convergence is uniform on those relatively closed subsets F of U satisfying $F \cap V_n \neq \emptyset$ only for finitely many n, then Theorem 3.1 is valid with $H^\infty(U)$ replaced by A and $H^\infty(0)$ replaced by $H_\infty(0) \cap A$.

Example of such an A. Let $U = \{z: |z| < 1\}$ and let $Q \subset \partial U$. Define A as those $f \in H^\infty(U)$ such that $\lim_{r\to 1} f(r\, e^{i\theta})$ exist whenever $e^{i\theta} \in Q$.

In the diameters of the components of the complement of X is bounded away from zero, and explicit construction of the set 0 in Theorem 3.1 can be carried out. This depends on some estimates of the analytic capacity and diameter of compact connected sets. (See Theorem 2.1 and Lemma 6.1 in Ch. VII of [4]).

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