PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 1, 1972

LACUNARITY FOR COMPACT GROUPS. II.

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Let G be a compact Abelian group with character group X. A standard characterization of Sidonicity of a subset P of X is given by the statement: if $f \in C(G)$ and the Fourier transform $\hat{f}$ vanishes on $X \backslash P$, then $\sum_{\chi \in X} |\hat{f}(\chi)| < \infty$. In this paper, we show that the characterization remains intact if $C(G)$ is replaced by any one of a large class of smaller function spaces on G. Extensions to compact non-Abelian groups are also given.

1. Introduction and notation. Throughout §§ 1–3, let G denote a compact Abelian group with character group X [we take all topological groups to be Hausdorff]. For any subset P of X and any subset E of $L^1(G)$, we will write E_P for the set of all f in E such that $\hat{f}(\chi) = 0$ for $\chi \in X \backslash P$. The space of functions in $L^1(G)$ with absolutely convergent Fourier series will be denoted by $A(G)$.

A subset P of X is said to be a *Sidon set* if $C_P(G) \subset A(G)$. The purpose of this paper is to describe a class of proper subsets E of $C(G)$ having the property that the inclusion

$$E_P \subset A(G) \tag{1}$$

implies [and is therefore equivalent to] the assertion

$$P \text{ is Sidon .} \tag{2}$$

It is well known ([4], (37.2)) that (2) implies [and is therefore equivalent to] the inclusion

$$L_P^\infty(G) \subset A(G) .$$

However, for $E \subsetneqq C(G)$, the equivalence of (1) and (2) does not appear to have been previously studied.

We now list some examples of sets $E \subset C(G)$ for which we can prove that (1) implies (2).

(a) For $p \in [1, \infty]$, we will write

$$A^p(G) = \{f \in C(G) \colon \hat{f} \in l^p(X)\} ,$$

so that $A^1(G) = A(G)$ and $A^p(G) = C(G)$ for $p \geqq 2$. If $1 < p < 2$ and $E = A^p(G)$, then (1) implies (2).

(b) For any w in $c_0(X)$, we define

$$A(G; w) = \{f \in C(G) \colon \hat{f}w \in l^1(X)\} .$$

Then (1) implies (2) for $E = A(G; w)$.

(c) The set E can be taken to be any countable intersection of spaces listed in (a) and (b). For example, E can be

$$A^{1+}(G) = \{f \in C(G): \hat{f} \in l^p(X) \quad \text{for all} \quad p > 1\} .$$

(d) Let G be the circle group T and let $E = U(T)$, the space of all f in $L^1(T)$ for which the sequence of symmetric partial sums of its Fourier series converges uniformly; see [5], p. 5. Then (1) implies (2). In this connection, it is interesting to note that Figà-Talamanca [3] has shown that the inclusion $C_P(T) \subset U(T)$ does *not* imply that P is a Sidon set.

In §4 we indicate how all of our results can be established for arbitrary compact groups.

We rely on the notation and terminology in [4] throughout this paper. We also write $PM(G)$ and $PF(G)$ for the spaces of pseudomeasures and pseudofunctions on G (see [5], pp. 27 and 44).

2. A general theorem.

Our first theorem is a quite general result giving certain sufficient conditions under which (1) implies (2), as in §1. The fourth hypothesis may look somewhat artificial, but in practice it is easy to apply and altogether works out to be natural.

THEOREM 2.1. *Let G be a compact Abelian group with character group X and let P be a subset of X. Suppose that E is a linear subspace of $C(G)$. The inclusion $E_P \subset A(G)$ implies that P is a Sidon set provided E satisfies the following four conditions*:

(i) $T_P(G) \subset E$;

(ii) *there is a topology t on E_P making (E_P, t) into a barrelled locally convex topological vector space* [see [1], p. 427];

(iii) *if $\chi \in P$, then $f \to \hat{f}(\chi)$ is a continuous function on (E_P, t)*;

(iv) *if L is a continuous linear functional on (E_P, t), then there exist a number α such that $0 < \alpha < 1$, and also $\mu \in M(G)$, $\psi \in l^\infty(X)$, and a finite subset Φ of P such that*

$$(1) \qquad |\psi(\chi)| \leqq \alpha \quad \text{for all} \quad \chi \in P \backslash \Phi$$

and

$$(2) \qquad L(f) = \mu * f(e) + \textstyle\sum_{\chi \in P} \psi(\chi)\hat{f}(\chi)$$

for all $f \in T_P(G)$.

Proof. We lose no generality in supposing that $E = E_P$. Thus we suppose that $E = E_P \subset A(G)$ and that conditions (i)—(iv) hold. By (37.2) in [4], to prove P a Sidon set, it suffices to consider any ϕ in $l^\infty(P)$ and to show that there is a measure $\nu \in M(G)$ satisfying

$$(3) \qquad |\phi(\chi) - \hat{\nu}(\chi)| \leqq \alpha \quad \text{for all} \quad \chi \in P\,,$$

where α is some number such that $0 < \alpha < 1$. Condition (iii) and the inclusion $E \subset A(G)$ show that

$$f \to \sum_{\chi \in P} |\phi(\chi)\hat{f}(\chi)| = \sup \{\sum_{\chi \in \Psi} |\phi(\chi)\hat{f}(\chi)| : \Psi \subset P, \Psi \text{ finite}\}$$

is a lower semicontinuous seminorm on E; by (ii) and [1], 7.1.1(a), this seminorm is therefore continuous on E. A fortiori, the linear functional L given by

$$L(f) = \sum_{\chi \in P} \phi(\chi)\hat{f}(\chi)$$

is continuous on E. For this L, select α, μ, ψ and Φ as in (iv). From (i) and (2) it follows that

$$\phi(\chi) = \hat{\mu}(\chi) + \psi(\chi) \quad \text{for all} \quad \chi \in P\,.$$

Define g as the trigonometric polynomial in $T_\Phi(G)$ such that $\hat{g}(\chi) = \phi(\chi) - \hat{\mu}(\chi)$ for $\chi \in \Phi$, and then set $\nu = \mu + g\lambda$, where λ denotes [normalized] Haar measure on G. Then ν belongs to $M(G)$ and (3) follows easily from (1).

REMARKS 2.2. (a) In (2.1.iv) we may also write $\psi = \hat{\sigma}$ for some pseudomeasure $\sigma \in PM(G)$, the sum in (2.1.2) then being just $\sigma * f(1)$.

(b) In all applications of (2.1) in the present paper, (2.1.ii) is satisfied by virtue of the fact that (E_P, t) is either a Banach space or a Fréchet space. It would be interesting to have some examples not of either of these types.

(c) Suppose that for $k \in \{1, 2, \cdots\}$, E_k is a subspace of $C(G)$ satisfying the conditions in Theorem (2.1), where $((E_k)_P, t_k)$ is a Fréchet space. Then

$$E = \bigcap_{k=1}^{\infty} E_k$$

satisfies these same conditions and so $E_P \subset A(G)$ implies that P is a Sidon set. [To see this, note that $E_P = \bigcap_{k=1}^{\infty} (E_k)_P$ and give E_P the topology t having a base at zero formed of intersections $U_1 \cap U_2 \cap \cdots \cap U_k \cap E_P$, where $k = 1, 2, \cdots$, and each U_j is a neighborhood of zero in $((E_j)_P, t_j)$. Equivalently, if $\{p_{k,n}: n = 1, 2, \cdots\}$ is a defining family of seminorms for the topology t_k, a defining family for t is obtained by taking finite sums of the restrictions to E_P of the $p_{k,n}$ with k and n varying.

A lengthy but routine argument shows that (E_P, t) is a Fréchet space and that (2.1.iii) holds. To verify (2.1.iv), use the Hahn-Banach theorem to prove that every continuous linear functional on (E_P, t) is a finite sum of restrictions to E_P of countinuous linear functionals on the various $((E_k)_P, t_k)$; [cf. [1], p. 147, Exercise 2.18].

3. Some applications of the general theorem.

(3.1) We list here some spaces to which Theorem (2.1) applies. The notation has been established in §1.

(a) For a fixed p in $(1,2)$, let $E = A^p(G)$. The topology t for E_P is defined by the norm

$$||f|| = ||f||_u + ||\hat{f}||_p .$$

It is easy to check that all the conditions of Theorem (2.1) are fulfilled. For example, condition (2.1.iv) is established by showing that if L is a continuous linear functional on (E_P, t), then there exist $\mu \in M(G)$ and $\psi \in l^{p'}(X)$ satisfying (2.1.2), p' being the exponent conjugate to p.

(b) For a fixed w in $c_0(X)$, let $E = A(G; w)$.
Note that for $P \subset X$, we have

$$E_P = \{f \in C_P(G): \hat{f}w|P \in l^1(P)\} .$$

The topology t on E_P is defined by the norm

$$||f|| = ||f||_u + ||w\hat{f}||_1 .$$

It is evident that (2.1.i)—(2.1.iii) are satisfied. Moreover, given a continuous linear functional L on (E_P, t), there exist $\mu \in M(G)$ and $\beta \in l^\infty(X)$ such that (2.1.2) holds with $\psi = w\beta$. Since ψ is in $c_0(X)$, (2.1.iv) is also satisfied (for any $\alpha > 0$).

(c) Remark (2.2.c) applied to (a) and (b) yields the following. Let w be in $c_0(X)$. In order that a subset P of X be Sidon, it is sufficient [and trivially necessary] that $\hat{f} \in l^1(X)$ for every f in $C_P(G) \cap A^{1+}(G) \cap A(G; w)$.

(d) The space $E = U(T)$ is a special case of a general class of spaces E to which (2.1) applies. To describe these spaces, we next establish some general notation and prove some general results.

NOTATION 3.2. Let I be a nonvoid set and S a topological linear space. If $\mathbf{s}: i \to s_i$ is a function from I into S and $s \in S$, we write

$$\lim_{(I)} \mathbf{s} = \lim_{(I)} s_i = s$$

if and only if to every neighborhood V of zero in S there corresponds a finite set $J \subset I$ such that

$$i \in I \backslash J \qquad \text{implies} \qquad s - s_i \in V .$$

[If I is the set of positive integers, this concept of limit is the usual one. However, if I is a directed set, this concept of limit is not in general the same as that usually associated with the convergence of the net $i \rightarrow s_i$, i.e., is not generally the same as asserting that to every V there corresponds an $i_0 \in I$ such that $s - s_i \in V$ for all $i \in I$ greater than i_0 in the given partial ordering of I.]

We write $c_0(I, S)$ for the linear space of all functions $\boldsymbol{s}$ from I into S such that $\lim_{(I)} s_i = 0$. Note that the range of every such function is a bounded subset of S. If S is a normed linear space [respectively, a Banach space], then $c_0(I, S)$ is a normed linear space [respectively, a Banach space] relative to the norm

$$|||\boldsymbol{s}||| = \sup_{i \in I} ||s_i|| .$$

The same observations apply to the linear space $l^1(I, S)$ of all functions $\boldsymbol{s}$ from I into S such that

$$|||\boldsymbol{s}|||_1 = \sum_{i \in I} ||s_i|| < \infty .$$

LEMMA 3.3. *Let I be a nonvoid set, let S be a normed linear space, and let S' denote the conjugate space of S. If L is a continuous linear functional on $c_0(I, S)$, there exists $\boldsymbol{\lambda} \in l^1(I, S')$ such that*

(i) $L(\boldsymbol{s}) = \sum_{i \in I} \lambda_i(s_i)$

for every $\boldsymbol{s} \in c_0(I, S)$.

Proof. By hypothesis, there exists $m \in (0, \infty)$ such that

$$|L(\boldsymbol{s})| \leqq m|||\boldsymbol{s}|||$$

for every $\boldsymbol{s} \in c_0(I, S)$. For $i \in I$ and $s \in S$, let $\boldsymbol{s}(i, s)$ denote the element of $c_0(I, S)$ defined by $\boldsymbol{s}(i, s)_i = s$ and $\boldsymbol{s}(i, s)_j = 0$ for $j \in I$ and $j \neq i$. Plainly we have $|||\boldsymbol{s}(i, s)||| = ||s||$. It follows that

$$\lambda_i(s) = L(\boldsymbol{s}(i, s))$$

defines an element of S'.

Now let J be any finite subset of I and suppose that $s_i \in S$ and that $||s_i|| \leqq 1$ for each $i \in J$. We then have

$$\sum_{i \in J} \lambda_i(s_i) = \sum_{i \in J} L(\boldsymbol{s}(i, s_i)) = L(\sum_{i \in J} \boldsymbol{s}(i, s_i))$$

and also

$$||| \sum_{i \in J} \boldsymbol{s}(i, s_i) ||| \leqq 1 .$$

On letting the s_i vary independently, we see that

$$\sum_{i \in J} ||\lambda_i|| \leqq m ,$$

and hence that $\boldsymbol{\lambda} = (\lambda_i)_{i \in I}$ belongs to $l^1(I, S')$. From this it follows at once that

$$L'(\boldsymbol{s}) = \sum_{i \in I} \lambda_i(s_i)$$

defines a continuous linear functional on $c_0(I, S)$. Moreover, as is easily verified, L and L' agree on the set of $\boldsymbol{s} \in c_0(I, S)$ having finite supports. Since this latter set is dense in $c_0(I, S)$, L and L' agree everywhere on $c_0(I, S)$ and so (i) is established.

THEOREM 3.4. *Let G be a compact Abelian group, let P be a subset of X, and let I be a nonvoid set. For each $i \in I$, let ζ_i be an element of $M(G) \cup PF(G)$. Suppose that*

(i) $m = \sup_{i \in I} \|\hat{\zeta}_i\|_\infty < \infty$,

and

(ii) $\lim_{(I)} \hat{\zeta}_i(\chi) = 0$ *for every* $\chi \in P$.

Then the set

$$E = \{f \in C_P(G): (\zeta_i * f)_{i \in I} \in c_0(I, C(G))\}$$

satisfies conditions (2.1.i)—(2.1.iv).[1] [Note that E depends upon both $(\zeta_i)_{i \in I}$ and P.]

Proof. We define a topology t on E by the norm

$$\|f\| = \|f\|_u + \sup_{i \in I} \|\zeta_i * f\|_u . \tag{1}$$

The definition of E automatically implies (2.1.i). Property (2.1.iii) obviously holds. A routine argument shows that E is complete for the norm (1); we omit the details. We need only verify (2.1.iv).

Let L be a continuous linear functional on E and α a positive number. We apply Lemma (3.3) with $S = C(G)$ and $S' = M(G)$. Thus there exist

$$\mu_1 \in M(G) \quad \text{and} \quad \boldsymbol{\lambda} = (\lambda_i)_{i \in I} \in l^1(I, M(G)) \tag{2}$$

such that

$$L(f) = \mu_1 * f(e) + \sum_{i \in I} \lambda_i * \zeta_i * f(e) \tag{3}$$

for all $f \in T_P(G)$. In view of (2), there is a finite subset J of I such that

$$m \sum_{i \in I \setminus J} \|\lambda_i\|_{M(G)} \leqq \frac{1}{2}\alpha . \tag{4}$$

Write I_1 for the set of i in I for which $\zeta_i \in M(G)$, and $I_2 = I \setminus I_1$, so

[1] Consider f in $C_P(G)$. If $\zeta_i \in M(G)$ for all i, then we automatically have $\zeta_i * f \in C(G)$ for all i, and so f belongs to E provided only that $\lim_{(I)} \|\zeta_i * f\|_u = 0$. If some ζ_i are not in $M(G)$, then $\zeta_i * f$ denotes the pseudofunction for which $(\zeta_i * f)^\wedge(\chi) = \hat{\zeta}_i(\chi)\hat{f}(\chi)$ for all $\chi \in X$ and does not automatically belong to $C(G)$. Thus in general, for f to be in E we must have $\zeta_i * f \in C(G)$ for all i and $\lim_{(I)} \|\zeta_i * f\|_u = 0$.

that $\hat{\zeta}_i \in c_0(X)$ for $i \in I_2$. Write also

$$\mu_2 = \mu_1 + \sum_{i \in J \cap I_1} \lambda_i * \zeta_i ,$$

$$\psi_1 = \sum_{i \in J \cap I_2} \hat{\lambda}_i \hat{\zeta}_i ,$$

$$\psi_2 = \sum_{i \in I \backslash J} \hat{\lambda}_i \hat{\zeta}_i .$$

Then (2) ensures that $\mu_2 \in M(G)$. Also, since J is finite, the definition of I_2 ensures that $\psi_1 \in c_0(X)$. By (i) and (4), we have

$$(5) \qquad \|\psi_2\|_\infty \leqq \frac{1}{2}\alpha.$$

From (3) we have for every $f \in T_P(G)$:

$$(6) \qquad L(f) = \mu_2 * f(e) + \sum_{\chi \in P} \psi_1(\chi)\hat{f}(\chi) + \sum_{\chi \in P} \psi_2(\chi)\hat{f}(\chi) .$$

Now choose a finite subset Φ_1 of X such that

$$(7) \qquad |\psi_1(\chi)| \leqq \frac{1}{2}\alpha \quad \text{for} \quad \chi \in X \backslash \Phi_1 ,$$

put $\Phi = \Phi_1 \cap P$, and define

$$\mu = \mu_2 + \sum_{\chi \in \Phi} \psi_1(\chi)\chi\lambda$$

and

$$\psi = \psi_1 \xi_{X / \Phi} + \psi_2 .$$

[ξ_A denotes the characteristic function of A.] Then $\mu \in M(G)$ and (6) gives for every $f \in T_P(G)$:

$$(8) \qquad L(f) = \mu * f(e) + \sum_{\chi \in P} \psi(\chi)\hat{f}(\chi) .$$

Since $P \backslash \Phi \subset X \backslash \Phi_1$, (7) shows that $|\psi_1(\chi)| \leqq (1/2)\alpha$ for $\chi \in P \backslash \Phi$; and (5) then goes to show that

$$\sup_{\chi \in P \backslash \Phi} |\psi(\chi)| \leqq \alpha .$$

This, coupled with (8), confirms (2.1.iv).

THEOREM 3.5. *Let G be a compact Abelian group, let P be a subset of X, and let I_k be a nonvoid set for $k \in \{1, 2, \cdots\}$. Suppose that*

(i) $\zeta_{i,k} \in M(G) \cup PF(G)$ *for* $i \in I_k$, $k \in \{1, 2, \cdots\}$;

(ii) $\sup_{i \in I_k} \|\hat{\zeta}_{i,k}\|_\infty < \infty$ *for* $k \in \{1, 2, \cdots\}$;

(iii) $\lim_{(I_k)} \hat{\zeta}_{i,k}(\chi) = 0$ *for* $\chi \in P$, $k \in \{1, 2, \cdots\}$.

Then

$$E = \{f \in C_P(G) \colon (\zeta_{i,k} * f)_{i \in I_k} \in c_0(I_k, C(G)) \quad \textit{for} \quad k = 1, 2, \cdots\}$$

has the property that the inclusion $E \subset A(G)$ is sufficient [and trivially necessary] *to ensure that P is Sidon.*

Proof. By (3.4), Theorem (2.1) applies to each

$$E_k = \{f \in C_P(G): (\zeta_{i,k} * f)_{i \in I_k} \in c_0(I_k, C(G))\} .$$

Since $E = \bigcap_{k=1}^{\infty} E_k$, (2.2.c) shows that (2.1) also applies to E.

(3.6) Some interesting special cases of (3.5) arise in the following manner. Suppose that P is countable and $\boldsymbol{P} = (P_i)_{i=1}^{\infty}$ is a sequence of finite subsets of P satisfying

$$P = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} P_i .$$

We take I to be the set of positive integers, select σ_k in $M(G) \cup PF(G)$ for $k \in \{1, 2, \cdots\}$, and define

$$\zeta_{i,k} = \sigma_k - \sum_{\chi \in P_i} \hat{\sigma}_k(\chi)\chi$$

for $i \in I$ and $k \in \{1, 2, \cdots\}$. Then (3.5.i)—(3.5.iii) are satisfied and

$$(1) \qquad \zeta_{i,k} * f = \sigma_k * f - s_{P_i}(\sigma_k * f) ,$$

where

$$s_{P_i} g = \sum_{\chi \in P_i} \hat{g}(\chi)\chi$$

is the P_i-partial sum of the Fourier series of g.

If we denote by $U(G, \boldsymbol{P})$ the set of $g \in C(G)$ such that $\lim_{i \to \infty} s_{P_i} g$ exists uniformly on G, the associated space E in (3.5) is now none other than

$$E = \{f \in C_P(G): \sigma_k * f \in U(G, \boldsymbol{P}) \quad \text{for} \quad k = 1, 2, \cdots\} .$$

In particular, if every σ_k is taken to be the unit mass at the identity element of G, E becomes

$$U_P(G, \boldsymbol{P}) = C_P(G) \cap U(G, \boldsymbol{P}) ,$$

which is the set of f in $C_P(G)$ whose Fourier series, grouped in the fashion specified by $\boldsymbol{P}$, converge uniformly on G.

COROLLARY 3.7. *A subset P of the group Z of integers is a Sidon set if and only if $U_P(T) \subset A(T)$.*

Proof. We have $U_P(T) = U(T, \boldsymbol{P})$ where $\boldsymbol{P} = (P_i)_{i=1}^{\infty}$ and $P_i = \{n \in P: |n| \leqq i\}$.

The next corollary illustrates how our results can be combined to obtain what appear to be very weak conditions sufficient to ensure that a set is a Sidon set.

COROLLARY 3.8. *Suppose that G is a compact Abelian group. Suppose that w is in $c_0(X)$, and that P, $\boldsymbol{P}$, and the σ_k are as in* (3.6). *In order that P be Sidon, it is sufficient* [and trivially necessary] *that $\hat{f} \in l^1(X)$ for every f in $C_P(G) \cap A^{1+}(G) \cap A(G; w)$ such that*

(i) $\sigma_k * f \in U(G, \boldsymbol{P})$ *for* $k = 1, 2, \cdots$.

Proof. Apply (2.2.c) and (2.1), taking into account (3.1.a), (3.1.b), and (3.6).

One of the standard inequalities known to be necessary and sufficient for P to be a Sidon set is [see [4], (37.2)] expressed by the existence of a number $\eta = \eta(P)$ such that

$$\|\hat{f}\|_1 \leqq \eta \|f\|_u$$

for every $f \in T_P(G)$. From Corollary (3.8) we will derive an apparently weaker inequality which is itself sufficient.

COROLLARY 3.9. *Suppose that w is in $c_0(X)$, and that P, $\boldsymbol{P}$ and the σ_k are as in* (3.6). *In order that P be a Sidon set, it is sufficient* [and trivially necessary] *that there exist $k \in \{1, 2, \cdots\}$, $p \in (1, \infty]$ and $\eta \in (0, \infty)$ such that*

(i) $\|\hat{f}\|_1 \leqq \eta\{\|f\|_u + \|\hat{f}\|_p + \|w\hat{f}\|_1 + \sum_{j=1}^k \sup_i \|s_{P_i}(\sigma_j * f)\|_u\}$

for every $f \in T_P(G)$.

Proof. Choose and fix a net (e_m) of elements of $T(G)$ such that

$$(1) \qquad \|e_m\|_1 \leqq 1 \quad \text{for every } m$$

and

$$(2) \qquad \lim_m \hat{e}_m(\chi) = 1 \quad \text{for every} \quad \chi \in P .$$

[The existence of (e_m) is proved in [4], p. 88, (28.53).] Consider any $f \in C_P(G) \cap A^{1+}(G) \cap A(G; w)$ satisfying (3.8.i). In view of (3.8), it will suffice to show that $\hat{f} \in l^1(X)$.

To this end, write $f_m = e_m * f$, so that $f_m \in T_P(G)$ for all m. From (1) we infer that

$$(3) \qquad \|f_m\|_u \leqq \|f\|_u, \|\hat{f}_m\|_p \leqq \|\hat{f}\|_p , \quad \text{and} \quad \|w\hat{f}_m\|_1 \leqq \|w\hat{f}\|_1$$

for every m. From (3.6.1) we have

$$s_{P_i}(\sigma_j * f_m) = e_m * s_{P_i}(\sigma_j * f) ,$$

and so

$$(4) \qquad \| s_{P_i}(\sigma_j * f_m) \|_u \leqq \| s_{P_i}(\sigma_j * f) \|_u$$

for every m, j, and i. Note that we have

$$(5) \qquad \sup_i \| s_{P_i}(\sigma_j * f) \|_u < \infty$$

for each j, since $\sigma_j * f \in U(G, \boldsymbol{P})$. Replacing f by f_m in (i) and noting (3), (4), and (5), we find that

$$\sup_m \| \hat{f}_m \|_1 < \infty \ .$$

This, combined with (2), shows that $\| \hat{f} \|_1 < \infty$, as we wished to show.

4. Extensions to arbitrary compact groups. In this section, G will denote an arbitrary compact group with dual object Σ. For any subset P of Σ and any $p \in \{0\} \cup [1, \infty]$, $\mathfrak{C}_p(P)$ is as defined in [4], (28.24) and (28.34). Fourier and Fourier-Stieltjes transforms are operator-valued functions defined on Σ as in [4], (28.34). The space of functions f on G with absolutely convergent Fourier series, i.e. $\hat{f} \in \mathfrak{C}_1(\Sigma)$, will be denoted by $A(G)$; this space is studied in [4], §34, and denoted there by $\mathfrak{R}(G)$.

The space $PM(G)$ of pseudomeasures on G is, by definition, the space of bounded linear functionals ζ on $A(G)$. For $\zeta \in PM(G)$, $\hat{\zeta}$ denotes the element in $\mathfrak{C}_\infty(\Sigma)$ for which

$$\zeta(f) = \sum_{\sigma \in \Sigma} d_\sigma tr(\hat{f}(\sigma)\hat{\zeta}(\sigma)^\sim) \quad \text{for} \quad f \in A(G) \ .$$

Thus $\zeta \rightarrow \hat{\zeta}$ is a linear isometry of $PM(G)$ onto $\mathfrak{C}_\infty(\Sigma)$. The space $PF(G)$ of pseudofunctions on G is all ζ in $PM(G)$ such that $\hat{\zeta} \in \mathfrak{C}_0(\Sigma)$. For further discussion about pseudomeasures on compact groups, see [4], (34.46).

The theorems and proofs in §§2 and 3 all carry over to this setting in a straightforward manner. We content ourselves with the following summary.

THEOREM 4.1. *Let G be a compact group with dual object Σ. A subset P of Σ is a Sidon set provided*

$$E_P = \{f \in E : \hat{f}(\sigma) = 0 \quad \textit{for all} \quad \sigma \in \Sigma \backslash P\} \subset A(G)$$

for any of the following spaces E:

(i) $A^p(G) = \{f \in C(G) : \hat{f} \in \mathfrak{C}_p(\Sigma)\}$, $p > 1$;

(ii) $A(G; F) = \{f \in C(G) : \hat{f}F \in \mathfrak{C}_1(\Sigma)\}$, $F \in \mathfrak{C}_0(\Sigma)$;

(iii) *any countable intersection of spaces listed in* (i) *and* (ii).

The space E can also be taken to be

$$\{f \in C_P(G) : (\zeta_{i,k} * f)_{i \in I_k} \in c_0(I_k, C(G)) \quad \textit{for} \quad k = 1, 2, \cdots\} \ ,$$

where

$$I_k \text{ is a nonvoid set for each } k \in \{1, 2, \cdots\} ;$$

$$\zeta_{i,k} \in M(G) \cup PF(G) \quad \text{for} \quad i \in I_k,\ k \in \{1, 2, \cdots\} ;$$

$$\sup_{i \in I_k} \|\hat{\zeta}_{i,k}\|_\infty < \infty \quad \text{for} \quad k \in \{1, 2, \cdots\} ;$$

$$\lim_{(I_k)} \hat{\zeta}_{i,k}(\sigma) = 0 \quad \text{for} \quad \sigma \in P,\ k \in \{1, 2, \cdots\} .$$

Added in proof. For a short proof of some of our results, see Ron C. Blei, *A note on some characterizations of Sidon sets*, to appear in Proc. Amer. Math. Soc.

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Received March 25, 1971. Research of the second author was supported by NSF Grant GP-19274. Research of the third author was supported by NSF Grant GP-20226.

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