PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 2, 1972

ON p-THETIC GROUPS

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The subject of this paper is a class of locally compact abelian (LCA) groups. Let p be a prime and let $Z(p^\infty)$ denote the group of complex p^nth roots of unity equipped with the discrete topology. An LCA group G is called *p-thetic* if it contains a dense subgroup algebraically isomorphic to $Z(p^\infty)$. It is shown that a p-thetic LCA group is either compact or is topologically isomorphic to $Z(p^\infty)$. This fact leads to the formulation of a property which characterizes the p-thetic, the monothetic, and the solenoidal groups. Applications to some purely algebraic questions are presented.

Let us take a paragraph to settle notation. Throughout, all groups are assumed to be LCA Hausdorff topological groups. Some LCA groups which we shall mention frequently are the integers Z taken discrete, the additive group Q of the rationals taken discrete, the additive group R of the real numbers with the usual topology, the circle T, the cyclic groups $Z(n)$ of order n, and the quasicyclic groups $Z(p^\infty)$, where p is a prime. Probably the most important group which we shall use is the group of p-adic integers, where p is a prime (see [2, §1] or [7, §10] for the definition and notation). The group of p-adic integers with its usual compact topology is written J_p; we use I_p to stand for the p-adic integers with the discrete topology. If G is an LCA group, then $\hat{G}$ stands for the character (or dual) group of G. In [7, 25.2] it is shown that the dual of J_p is $Z(p^\infty)$. If G is a group, we let $B(G)$ denote the torsion subgroup of G, while $B_p(G)$ denotes the set of elements of G whose order is a power of a fixed prime p. Topological isomorphism is denoted by $\cong$.

THEOREM 1. *Let G be a p-thetic LCA group. Then either G is compact or else G is topologically isomorphic to $Z(p^\infty)$.*

Proof. Since G is p-thetic, there is a continuous homomorphism $f: Z(p^\infty) \to G$ having dense image. Hence the transpose map $f^*: \hat{G} \to J_p$ is one-one [7, 24.41]. We wish to show that either $\hat{G}$ is discrete or $\hat{G} \cong J_p$. We first note that $\hat{G}$ must be totally disconnected, since f^* is one-one and J_p is totally disconnected. Thus $\hat{G}$ contains a compact open subgroup U. If U is trivial, then $\hat{G}$ is discrete. Otherwise, $f^*(U)$ is a nontrivial compact subgroup of J_p and is hence open in J_p [7, 10.16(a)]. Now the restriction of f^* to the compact subgroup U is a topological isomorphism from U onto the open subgroup $f^*(U)$

of J_p. Hence f^* is an open mapping, so that $\hat{G}$ is topologically isomorphic to $f^*(\hat{G})$. Since every closed subgroup of J_p is topologically isomorphic to J_p itself, we conclude that $\hat{G} \cong J_p$. This completes the proof.

Now let G and H be LCA groups. We say that G is *H-dense* if there exists a continuous homomorphism $f: H \to G$ such that $f(H)$ is a dense subgroup of G. Thus the monothetic groups are the Z-dense groups, the solenoidal groups are the R-dense groups, and the p-thetic groups are just the $Z(p^\infty)$-dense groups. As is well known, the LCA monothetic and solenoidal groups are either compact, or else topologically isomorphic to Z and R, respectively [7, 9.1]. As we have just proved, a p-thetic LCA group is either compact or is topologically isomorphic with $Z(p^\infty)$. These facts lead us to the very natural question: For which LCA groups H is it the case that every H-dense LCA group G is either compact or is topologically isomorphic to H? Since every H-dense group G is automatically compact for compact H, the question is of interest only for noncompact H. It is not difficult to determine the answer to this question, and our answer will show that, in a sense, the study of the p-thetic groups complements the study of the monothetic and solenoidal groups.

THEOREM 2. *Let H be a non-compact LCA group. The following are equivalent:*

(1) *Every H-dense LCA group G is either compact or is topologically isomorphic to H.*

(2) *H is topologically isomorphic with either Z, R, or $Z(p^\infty)$, where p is a prime.*

Proof. We have already shown that (2) $\to$ (1). For the converse, assume that (1) holds for H. We show that any strictly stronger topology on $\hat{H}$ which makes $\hat{H}$ into a locally compact group must be the discrete topology. To this end, let D denote $\hat{H}$ with a strictly stronger locally compact topology. Then the identity map $i: D \to \hat{H}$ is continuous and one-one, so that the transpose map $i^*: H \to \hat{D}$ has dense image [7, 24.41]. Since (1) holds, either $\hat{D} \cong H$ or else $\hat{D}$ is compact. Since the first alternative has been ruled out, we conclude that D is discrete, as we wished to show. We now invoke [9, Theorem 2] or [10, Theorem 2.1] to conclude that $\hat{H}$ contains an open subgroup U which is topologically isomorphic with either T, R or J_p for some prime p. Hence $\hat{U}$ is a quotient H by a closed subgroup. If $\pi: H \to \hat{U}$ is the projection of H onto $\hat{U}$, we conclude from (1) that either $H \cong \hat{U}$ or else $\hat{U}$ is compact. Since $\hat{U}$ is not compact, we conclude that $H \cong \hat{U}$, so that $H \cong Z$, $H \cong R$, or $H \cong Z(p^\infty)$. Thus (1) $\Rightarrow$ (2), which completes the proof.

Since a compact group is p-thetic if and only if its discrete dual is isomorphic to a subgroup of the discrete group I_p of p-adic integers, we will do well, before mentioning some examples and simple properties of p-thetic groups, to recall a few basic properties of the group I_p, all of which may be found in [4] and [5]. The group I_p is a reduced, torsion-free group of cardinality (and hence rank) of the power of the continuum. It contains an isomorphic copy of the group Q_p consisting of all rational numbers with denominators prime to p. The group I_p contains no elements of infinite p-height, but every element has infinite q-height if q is a prime different from p (we say that an element x in an additively written group G has infinite p-height if the equation $p^n y = x$ can be solved for y in G for an arbitrary positive integer n).

We now mention a few examples of p-thetic groups. The circle T is p-thetic for all primes p. In fact, since I_p has rank the power of the continuum, it contains isomorphic copies of the free abelian group of rank M if M does not exceed the power of the continuum. Thus the torus T^M is p-thetic for all p if and only if M does not exceed the power of the continuum. Other examples of p-thetic groups are $\hat{Q}_p$ and $\hat{I}_p$. These groups are p-thetic for only the one prime p. The group $\hat{I}_p$ (which is the Bohr compactification of $Z(p^\infty)$) is the "largest compact p-thetic group" in the sense that every compact p-thetic group (where p is a fixed prime) is a quotient of $\hat{I}_p$ by a closed subgroup.

Every compact p-thetic group is a connected monothetic group [7, 25.13] and is hence solenoidal [7, 25.14]. Obviously, the torsion subgroup of a p-thetic group is dense in the group, but it is easy to give examples of compact solenoidal groups with dense torsion subgroup which are not p-thetic for any prime p. For example, let G be the dual of the direct sum (taken discrete) of the groups Q_p and Q_q, where p and q are distinct primes. It is easy to see that G could not be isomorphic to a subgroup of a p-adic integer group (see the remarks above about p-height), and the fact that G has dense torsion subgroup follows from [8, Theorem 2] or [1, Proposition 7].

Professor L. Fuchs has kindly informed one of the authors that, to the best of his knowledge, necessary and sufficient conditions for a group to be embeddable in I_p are unknown. Therefore we are unable to give intrinsic characterizations of the p-thetic groups, as we can for the monothetic and solenoidal groups (in terms of weight, rank, etc.). The remainder of this paper will be concerned with certain special p-thetic groups and their application to the theory of infinite abelian groups.

THEOREM 3. *Let G be a compact connected group of dimension one. Then either $G \cong \hat{Q}$ or else G is p-thetic for some prime p.*

Proof. If G is torsion-free it follows from [7, 24.28 and 25.8] that $G \cong \hat{Q}$. Otherwise G contains an isomorphic copy H of $Z(p^\infty)$ for some prime p, by the structure theorem for divisible groups [7, A. 14] and the fact that a connected LCA group is divisible [7, 24. 24]. We shall show that the closure $\bar{H}$ of H is dense in G. Since H is divisible, it follows that every non-trivial continuous character of $\bar{H}$ has infinite range, so that $(\hat{\bar{H}})$ is torsion-free. But $(\hat{\bar{H}}) \cong \hat{G}/A(\hat{G}, H)$, where $A(\hat{G}, H)$ is the annihilator of H in $\hat{G}$ (see [7, 24. 5]). Since every proper quotient of a subgroup of Q is a torsion group, and since every group of rank one is isomorphic to a subgroup of Q [7, A. 16], it follows that $A(\hat{G}, H) = \{1\}$, so that $\bar{H} = G$, and therefore G is p-thetic.

REMARK 1. The group G in Theorem 3 may be p-thetic for all p, e.g. $G = T$. However, the circle is not the only one-dimensional compact group which is p-thetic for all p. For example, let us define a subgroup H of Q in the following way. Let p_n denote the nth prime and let H_n denote the set of rational numbers of the form $k/(p_1 p_2 \cdots p_n)$, where k is an integer. The sets H_n define an ascending sequence of subgroups of Q. If we let H be the union of the H_n, then we can show that H is isomorphic to a subgroup of Q_p for each p, but that H is not isomorphic to Z. Thus if we set $G = \hat{H}$, we have an example of a one-dimensional compact group which is p-thetic for all p but is not isomorphic to T.

Before proceeding to our next results, we review briefly the concepts of purity and p-purity. If G is a group and n a positive integer, we write nG for the set of elements of G of the form nx, where x is in G. A subgroup H of a group G is called *pure* if and only if $nH = H \cap nG$ for each positive integer n and *p-pure* if and only if $p^n H = H \cap p^n G$ for each positive integer n, where p is a prime. It is easy to see that if G is torsion-free, a subgroup H is pure (respectively, p-pure) if and only if G/H is torsion-free (respectively, $B_p(G/H) = \{0\}$).

DEFINITION 1. Let G be a compact p-thetic group. We say that G is *pure p-thetic* if and only if $B(G) \cong Z(p^\infty)$ and that G is *p-pure p-thetic* if and only if $B_p(G) \cong Z(p^\infty)$.

Before proceeding to justify the use of the terminology of this definition, we need to state a lemma.

LEMMA 1. *Let H be a p-pure subgroup of I_p. Then the index of pH in H is p.*

Proof. First note that since H has no elements of infinite p-

height, $pH \subsetneqq H$. Let $x = (x_0, x_1 \cdots)$ be an element in H but not in pH. Note that $x_0 \neq 0$, since otherwise x would be in pI_p and hence in pH, since H is *p-pure*. We claim that the coset $x + pH$ is a generator of the quotient group H/pH, so that $H/pH \cong Z(p)$. To see this, let $w + pH$ be an element of H/pH, where $w = (w_0, w_1, \cdots)$ is in H. Let y_i denote the first coordinate of ix, for $0 \leqq i \leqq p-1$. Then $w_0 = y_i$ for some i between 0 and $p-1$. Hence $w - ix$ has 0 in its first coordinate, so that $w - ix$ is in pH. That is, $w + pH = i(x + pH)$, which completes the proof.

THEOREM 4. *Let G be compact and let p be a fixed prime. The following are equivalent*:

(1) *G is pure p-thetic,*

(2) *$\hat{G}$ is isomorphic to a pure subgroup of I_p.*

Proof. Assume (1). Since G is p-thetic, there is a subgroup H of I_p such that $\hat{G} \cong H$. Let G_n denote the subgroup of elements of G having order n. By (1) it follows that $G_p \cong Z(p)$ and that G_q is trivial for all primes $q \neq p$. We conclude from [7, 24. 22] that $H/pH \cong Z(p)$ and that $qH = H$ for all primes $q \neq p$. Let us assume, for the moment, that there is an element $x = (x_0, x_1, \cdots)$ in H with $x_0 \neq 0$. In this case, we show that H is pure in I_p. Clearly, it suffices to show that $H \cap p^n I_p = p^n H$. First, suppose that $py \in H$ for some y in I_p. Since $H/pH \cong Z(p)$, we have that the coset $x + pH$ is a generator of H/pH. Thus $py + pH = ix + pH$ for some i between 0 and $p-1$. Hence there exists z in H such that $py = ix + pz$, so that $ix = p(y - z)$. This means that ix has 0 in its first coordinate. This can occur only if $i = 0$, so that $y = z$, and hence y is in H. This proves that $H \cap pI_p = pH$. That $H \cap p^n H = p^n H$ for all positive n follows by a simple induction argument. Thus, in this case, H is pure in I_p.

Finally, to show that the assumption about x may always be made, we need only consider an appropriate subgroup L_k of I_p, where L_k consists of all sequences $x = (x_0, x_1, \cdots)$ in I_p with $x_n = 0$ for n less than k, and use the fact that $L_k \cong I_p$. This completes the proof that (1) $\Rightarrow$ (2).

Conversely, assume (2). Let H be a pure subgroup of I_p such that $\hat{G} \cong H$. Then G is p-thetic, and it remains only to show that $B(G) \cong Z(p^\infty)$. By Lemma 1, $H/pH \cong Z(p)$, since a pure subgroup is automatically p-pure. Hence $G_p \cong Z(p)$, by [7, 24. 22]. Similarly, since $qH = H$ for all primes $q \neq p$ (since H is pure in I_p), it follows that G_q is trivial for $q \neq p$. Hence $B(G) \cong Z(p^\infty)$, so that G is pure p-thetic, i.e. (2) $\Rightarrow$ (1).

REMARK 2. The authors of [6] (see [4, Exercise 24 on p. 202]) show, without use of duality, that a reduced torsion-free group H has a unique maximal subgroup if and only if H is isomorphic to a pure subgroup of some group I_p. This can be deduced from Theorem 4 above in the following way. Let H be as indicated. It follows from [8, Theorem 2] or [1, Proposition 7] that $B(G)$ is dense in G, where $G = \hat{H}$. Since G must have unique minimal closed subgroup, and since $B(G)$ is divisible, it follows that $B(G) \cong Z(p^\infty)$ for some prime p, so that G is pure p-thetic. Hence H is isomorphic to a pure subgroup of I_p by Theorem 4. The converse is straightforward. Of course, it should be pointed out, going in the contrary direction, that our Theorem 4 can be deduced, via duality, from the result mentioned in [6].

THEOREM 5. *Let G be compact and let p be a fixed prime. The following are equivalent:*

(1) *G is p-pure p-thetic,*

(2) *$\hat{G}$ is isomorphic to a p-pure subgroup of I_p.*

Proof. The proof of the implication (1) $\Rightarrow$ (2) follows along the same lines as the corresponding proof in Theorem 4, so that we omit it. Next, assume (2). Thus G is p-thetic, and it only remains to show that $B_p(G) \cong Z(p^\infty)$. But this follows from Lemma 1, as in the proof of Theorem 4. Hence (2) $\Rightarrow$ (1), completing the proof.

REMARK 3. In [2] Armstrong has shown, by a study of the extensibility of endomorphisms of p-pure subgroups of I_p, that a p-pure subgroup of I_p must be indecomposable. We can provide an altogether different proof of this fact by using Theorem 5 above. We need only observe that a p-pure p-thetic group G cannot be written as the the topological direct sum of two of its proper closed subgroups, since each summand would be p-thetic, whereas $B_p(G) \cong Z(p^\infty)$.

In closing, we mention a criterion for a compact connected group to be p-pure p-thetic. This criterion is a direct translation, via duality, of a theorem due to Armstrong (see [3, Proposition 2]).

PROPOSITION 1. *Let G be compact and connected, and let p be a fixed prime. Then G is p-pure p-thetic if and only if*

(1) *$B_p(G)$ is dense in G, and*

(2) *G is topologically indecomposable and G/H is topologically indecomposable for every closed subgroup H of G such that $pH = H$.*

Proof. This follows by duality from Armstrong's result mentioned above and the fact that if H is a torsion-free abelian group, then a subgroup U of H is p-pure if and only if its annihilator in $\hat{H}$ is p-divisible.

REMARK 4. It follows from the above proposition that the p-thetic group G defined in Remark 1 is p-pure p-thetic for each prime p, since condition (1) holds, as shown in Remark 1, and condition (2) follows from the fact that G is of dimension one, so that it and all its quotients are topologically indecomposable.

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Received February 25, 1971.

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