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THE INFLATION—RESTRICTION THEOREM FOR AMITSUR COHOMOLOGY

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In this paper we develop a generalization of the classical exactness of the inflation—restriction sequence in group cohomology. Our main theorems relate the Amitsur cohomology of algebras to that of subalgebras.

1. Introduction. Throughout, R is a commutative ring, unadorned $\otimes$ means tensor product over R, all algebras are commutative, and if S is an R-algebra, S^j denotes the tensor product $S \otimes \cdots \otimes S$, j times. R-Alg and Ab denote the categories of commutative R-algebras and abelian groups, respectively.

For any R-algebra S there are R-algebra maps $\varepsilon_i^n \colon S^n \to S^{n+1}$ given by $\varepsilon_i^n(s_0 \otimes \cdots \otimes s_{n-1}) = s_0 \otimes \cdots \otimes s_{i-1} \otimes 1 \otimes s_i \otimes \cdots \otimes s_{n-1}$, $i = 0, 1, \cdots, n+1$. These are called the (n-dimensional) *co-face maps for S/R.* Generally the superscript will be suppressed. The co-face maps are easily seen to satisfy the co-face relations:

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1}\varepsilon_i \quad \text{for} \quad i \leqq j .$$

If $F \colon R\text{-Alg} \to Ab$ is any functor, the Amitsur cochain complex, $C(S/R, F)$, is defined by $C^n(S/R, F) = F(S^{n+1})$, $n = 0, 1, 2, \cdots$ [1, 2, 6]. The coboundary operator $d^n \colon F(S^{n+1}) \to F(S^{n+2})$ is given by $d^n = \sum_{i=0}^{n+1}(-1)^i F(\varepsilon_i)$. It is a consequence of the co-face relations that a complex results, i.e., that $d^{n+1}d^n = 0$. The homology $\operatorname{Ker} d^n/\operatorname{Im} d^{n-1}$ of this complex is the *Amitsur cohomology of S/R with coefficients in F*, denoted $H^n(S/R, F)$. As usual, $H^0(S/R, F) = \operatorname{Ker} d^0$.

Let $F_1 \colon R\text{-Alg} \to Ab$ be another functor and let $\eta \colon F \to F_1$ be a natural transformation. Then $C(1, \eta) = \eta_{S^{n+1}} \colon F(S^{n+1}) \to F_1(S^{n+1})$ is a map of complexes and so induces a map $H^n(1, \eta) \colon H^n(S/R, F) \to H^n(S/R, F_1)$.

We say a sequence $0 \to F^{\omega} F_1 \chi F_2 \to 0$ is exact if $0 \to F(A) \xrightarrow{\omega_A} F_1(A) \xrightarrow{\chi_A} F_2(A) \to 0$ is an exact sequence of abelian groups for each R-algebra A. Indeed the usual long sequence results from a short exact sequence of coefficients.

THEOREM 1.1. [6, p. 47]. *Let $0 \to F \xrightarrow{\omega} F_1 \xrightarrow{\chi} F_2 \to 0$ be an exact sequence of functors. Then there are maps $\delta^n(S)$ making*

$$\begin{aligned}\cdots \longrightarrow H^{n-1}(S/R, F_2) \xrightarrow{\delta^{n-1}(S)} H^n(S/R, F) \xrightarrow{H^n(1,\omega)} H^n(S/R, F_1)\\ \xrightarrow{H^n(1,\chi)} H^n(S/R, F_2) \xrightarrow{\delta^n(S)} H^{n+1}(S/R, F) \longrightarrow \cdots\end{aligned}$$

exact and this sequence is natural in S.

This is a standard result, a consequence of the fact that $0 \to F(S^{n+1}) \xrightarrow{\omega_{S^{n+1}}} F_1(S^{n+1}) \xrightarrow{\chi_{S^{n+1}}} F_2(S^{n+1}) \to 0$ is a short exact sequence of complexes.

REMARK. The entire discussion thus far in no way depends upon F being defined on all of R-Alg. If A is a full subcategory of R-Alg containing S^n and T^n and F, F_1 are abelian group valued functors on A, then all the preceding material is still valid.

2. Inflation-restriction. By an *(R-based) Grothendieck Topology* T (cf. [7]) we mean a category, Cat T, of commutative R-algebras and a collection, Cov T, of families called *covers* $\{U \to U_i\}$ of morphisms satisfying axioms dual to those of [3, pp. 1–2]. (In particular, fiber products are replaced by tensor products.) With this convention a *presheaf*, F (of abelian groups) is simply a functor Cat $T \to Ab$ and a presheaf F is a *sheaf* if for every cover $\{U \to U_i\}$, the induced diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \otimes_U U_j)$$

is an equalizer diagram (equivalently, the natural map $F(U) \to H_T^0(\{U \to U_i\}, F)$ [3, I, Sec. 3] is an isomorphism).

REMARK 3.1. If $A \to B$ is a map in Cat T and F a presheaf, the complex yielding $H_T^n(\{A \to B\}, F)$ coincides with the Amitsur complex. The definition of sheaf may be phrased as follows: the natural maps $F(S) \to H^0(T/S, F)$ and $F(S^2) \to H^0(T^2/S^2, F)$ are isomorphisms and $F(S^3) \to H^0(T^3/S^3)$ is a monomorphism. Among others which we examine in the next section, the functor which assigns to each R-algebra its multiplicative group of units satisfies this hypothesis if S/R and T/S are faithfully flat. (This follows, for example, from [6, Lemma 3.8].)

Another cohomology theory is defined as follows: the category $\mathscr{S}$ of sheaves on T is abelian with enough injectives [3, Ch. II, Thm. 1.6 (i) and 1.8 (i)]. For any object U in Cat T, the evaluation functor $E_V\colon \mathscr{S} \to Ab$ given by $E_V(F) = F(V)$ is left exact [3, Ch. II, Thm. 1.8 (iii)]. The nth right derived functor of E_V is denoted $H_T^n(V, -)$ and the group $H_T^n(V, F)$ is called the nth Grothendieck cohomology group of V with coefficients in F.

Let $R \xrightarrow{i} S \xrightarrow{j} T$ and $\underline{A}$ be a full subcategory of R-Alg which is closed under tensor products. Regard S and T as R-algebras and T as S-algebra via i, ij, and j respectively.

The map $H^n(j, 1)\colon H^n(S/R, F) - H^n(T/R, F)$ induced by j is called inflation and denoted *inf*.

Now i induces an R-algebra map $T^n \to T \otimes_S \cdots \otimes_S T$ given by $t_1 \otimes \cdots \otimes t_n \to t_1 \otimes_S \cdots \otimes_S t_n$. This is easily seen to commute with the face maps and so induces a map of complexes and in turn a map of cohomology $H^n(T/R, F) \to H^n(T/S, F)$, called restriction and denoted *res*.

Note that if $A \to B$ is a map in $\underline{A}$, then so is the multiplication map $B \otimes_A B \to B$ given by $x \otimes y \to xy$, being simply the composition $B \otimes_A B \to B \otimes_B B \simeq B$.

Our main theorem is

THEOREM 3.2. (Exactness of the inflation—restriction sequence) *Let X be a Grothendieck topology whose category $\underline{A}$ is such that $\{i\}$ and $\{j\}$ are covers. If F is a sheaf on X, then the inflation—restriction sequence*

$$0 \longrightarrow H^n(S/R, F) \xrightarrow{\text{inf}} H^n(T/R, F) \xrightarrow{\text{res}} H^n(T/S, F)$$

is exact if $n = 1$. Suppose $n \geqq 1$ and let Σ be the set of algebras S^i, T^i, or $T \otimes_S \cdots \otimes_S T$ (i times), $i \leqq n + 1$. If $H^j_X(A, F) = 0$ for all $j < n$ and for all A in Σ, then the inflation—restriction sequence is exact for n.

Proof. We induce on n.

The case $n = 1$ can be deduced from the spectral sequence of Cech cohomology [3, Ch. II, (3.1)] but a tedious direct argument can be given mimicing the corresponding proof for group cohomology [4, Ch. IV, Sec. 5, Prop. 5]. We illustrate the proof that inf. is a monomorphism:

Consider the diagram whose rows are exact since F is a sheaf:

$$\begin{tikzcd}
0 \arrow[r] & F(S \otimes_K S) \arrow[r, "F(j \otimes j)"] \arrow[d, dashed, shift right, "F(M_S)"'] & F(T \otimes_R T) \arrow[d, dashed, shift right, "F(M_T)"'] \arrow[dr, "F(\rho)"] & \\
0 \arrow[r] & F(S) \arrow[r, "F(j)"] \arrow[u, shift right, "d'^0"'] & F(T) \arrow[r, "F(\bar{\varepsilon}_0) - F(\bar{\varepsilon}_1)"] \arrow[u, shift right, "d^0"'] & F(T \otimes_S T)
\end{tikzcd} \tag{1}$$

with $d'^0 = F\varepsilon'_0 - F\varepsilon'_1$, $d^0 = F\varepsilon_0 - F\varepsilon_1$, with ε'_i and ε_i the face maps for S/R and T/R respectively and where $\rho(x \otimes_R y) = x \otimes_S y$, M_S and M_T are the multiplication maps from $S \otimes S$ to S and $T \otimes T$ to T respectively.

The solid arrows of the diagram clearly commute, the commutativity of the square being an example of an R-algebra map inducing a map of complexes.

If x in $F(S \otimes S)$ is a one cocycle whose cohomology class gets mapped to zero by inf, then $F(j \otimes j)\ (x) = (F(\varepsilon_0) - F(\varepsilon_1))\ (y)$ for some y in $F(T)$. We must show that the cohomology class of x was already zero, i.e., that there is an element z in $F(S)$ such that $(F(\varepsilon'_0) - F(\varepsilon'_1))\ (z) = x$. By commutativity of solid arrows and exactness of the rows in (1), it clearly suffices to show that $(F(\bar{\varepsilon}_0) - F(\bar{\varepsilon}_1))\ (y) = 0$. But by the definition of y and the commutativity of (1), this is the same as establishing

$$F(\rho)F(j \otimes j)(x) = 0\ . \tag{2}$$

Now in (1) the square with the dotted arrows clearly commutes as does

$$\begin{tikzcd} T \arrow[r, "\varepsilon_0"] \arrow[d, "\varepsilon_1"'] & T \otimes T \arrow[d, "M_T"] \\ T \otimes_R T \arrow[r, "M_T"] & T \end{tikzcd} \tag{3}$$

so that

$$F(j)F(M_S)(x) = F(M_T)F(j \otimes j)(x) = F(M_T)(F(\varepsilon_0) - F(\varepsilon_1))(y)$$

(by hypothesis on x)

and this is zero by the commutativity of (3). But since F is a sheaf, the map $F(j)$ is monic so we have

$$F(M_S)(x) = 0\ . \tag{4}$$

Finally if $\lambda\colon S \to T \otimes_S T$ is given by $\lambda(s) = s \otimes 1 = 1 \otimes s$ we clearly have $\rho(j \otimes j) = \lambda M_S$ as maps from $S \otimes_R S \to T \otimes_S T$ and multiplying (4) by $F(\lambda)$ shows

$$0 = F(\lambda)F(M_S)(x) = F(\rho)F(j \otimes j)(x)\ .$$

Thus (2) is established, completing the proof that inf is monic.

The remainder of the case $n = 1$ is proved by similar arguments.

For the induction we will make a "dimension shifting" argument. Let $n > 1$. Choose an injective sheaf F^* and a sheaf F' so that

$$0 \longrightarrow F \longrightarrow F^* \longrightarrow F' \longrightarrow 0$$

is exact in $\mathscr{S}$. Now in general this is not exact at each A in $\underline{A}$ so we can not immediately derive a long exact sequence of Amitsur cohomology.

However, since $H^1_X(A,\ )$ is the derived functor of "evaluation at A" we have an exact sequence

$$0 \longrightarrow F(A) \longrightarrow F^*(A) \longrightarrow F'(A) \longrightarrow H^1_X(A, F) .$$

By hypothesis the last term is 0 for all A in Σ.

Consequently we get an exact (up to dimension n) sequence of Amitsur cochain groups

$$0 \longrightarrow C^i(S/R, F) \longrightarrow C^i(S/R, F^*) \longrightarrow C^i(S/R, F') \longrightarrow 0 \quad i \leqq n$$

and similar sequences upon replacing S/R by T/R and T/S respectively.

In the usual fashion (cf. Thm. 1.1), these induce exact columns in the following diagram

$$\begin{array}{ccccccc}
 & & H^n(S/R, F^*) & \xrightarrow{\text{inf}} & H^n(T/R, F^*) & \xrightarrow{\text{res}} & H^n(T/S, F^*) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & H^n(S/R, F) & \xrightarrow{\text{inf}} & H^n(T/R, F) & \xrightarrow{\text{res}} & H^n(T/S, F) \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
0 & \longrightarrow & H^{n-1}(S/R, F') & \xrightarrow{\text{inf}} & H^{n-1}(T/R, F') & \xrightarrow{\text{res}} & H^{n-1}(T/S, F') \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & H^{n-1}(S/R, F^*) & \xrightarrow{\text{inf}} & H^{n-1}(T/R, F^*) & \xrightarrow{\text{res}} & H^{n-1}(T/S, F^*) .
\end{array}$$

That this diagram commutes is an immediate consequence of definitions of inf and res.

Now F^* is injective and so Cech cohomology of any cover with coefficients in F^* vanishes [3, Prop. 4.3 (iv), p. 40]. But again using Remark 3.1, we conclude that in the above diagram all the Amitsur cohomology with coefficients in F^* also vanishes.

Hence the maps δ are isomorphisms and the desired exactness will follow by induction if we can show that $H^j_X(A, F') = 0$ for all A in Σ and for all $1 \leqq j \leqq n - 1$. But this is immediate: F^* being injective, the short exact sequence $0 \to F \to F^* \to F' \to 0$ of sheaves yields $H^j_X(A, F') \cong H^{j+1}_X(A, F) = 0$ for all $1 \leqq j < n - 1$ and for all A in Σ.

This completes the proof of the theorem.

The full strength of the definition of sheaf is in fact not needed in case $n = 1$ in the above theorem. All that is required is that the sheaf property hold on $\{S^n \to T^n\}$, $n = 1, 2, 3$, however in practice the functors of interest which are not sheaves do not even satisfy this.

4. Etale sheaves and group cohomology. In this section we briefly sketch how the classical inflation—restriction theorem for group cohomology can be recovered from our results by use of the étale topology.

Let G be a finite group, H a normal subgroup and choose fields $k \subseteqq L$ with $G = \mathrm{Gal}\,(L/k)$. Let $N = L^H$ be the fixed field of H and let A be any G module. By a straight-forward modification of the results of I, Sec. 4 and 5 of [7] (cf. "Supplements" in [7]) one can show that there is a topology $T = T_{L/k}$, analogous to the usual étale topology, which has the following properties: (1) Every sheaf on T is additive [9, p. 9]. (2) The category $\mathscr{S}$ of sheaves on T is naturally equivalent to the category of G-modules. This equivalence associates to any sheaf F the module $F(L)$ with g in G acting as $F(g)$. If A is a module and M a subfield of L normal over k (all such subfields are among the objects of Cat T) then the sheaf F_A associated to A has $F_A(M) = A^{M'}$ where M' is the subgroup of G which fixes M.

With these observations one can prove the classical group cohomology theorem:

THEOREM 4.1. [4, Ch. IV, Sec. 5, Prop. 5] *Let G be a finite group, H a normal subgroup and A a G-module. Then*

$$0 \longrightarrow H^n(G/H, A^H) \xrightarrow{\text{inf}} H^n(G, A) \xrightarrow{\text{res}} H^n(H, A)$$

is exact for $n = 1$. If $H^i(H, A) = 0$ for $1 \leqq i < n$, then the sequence is exact for n.

Proof. Letting F be the sheaf associated to A one deduces from [5, Thm. 5.4] a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^n(N/k, F) & \xrightarrow{\text{inf}} & H^n(L/k, F) & \xrightarrow{\text{res}} & H^n(L/N, F) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^n(G/H, F(L)^H) & \xrightarrow{\text{inf}} & H^n(G, F(L)) & \xrightarrow{\text{res}} & H^n(H, F(L)) \end{array}$$

with each vertical map an isomorphism. It thus suffices to show the exactness of the upper sequence.

Since N/k, L/k and L/N are Galois, the set Σ of Theorem 3.2 consists of algebras which are the products of copies of N or copies of L.

The arguments of the Supplements, of [7] show that $H^n_T(X, F) \cong H^n(\mathrm{Gal}\,(L/X), F(L))$ for $X = N$ or L. Since sheaves on T are additive, dimension shifting shows $H^n_T(A \times B, F) \cong H^n_T(A, F) \oplus H^n_T(B, F)$ for any algebras A and B in Cat T. It then follows that the hypotheses of Theorem 3.2 reduce to requiring $H^j(H, A) = 0$ and $H^j(\mathrm{Gal}\,(L/L), A) = 0$ for $j < n$. The latter is trivial and the former is assumed, completing the proof.

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