PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 2, 1972

SOME H^p SPACES WHICH ARE UNCOMPLEMENTED IN L^p

SAMUEL E. EBENSTEIN

Let T^j denote the compact group which is the Cartesian product of j copies of the circle where j is a positive integer or ω. If $1 \leqq p \leqq \infty$ let $L^p(T^j)$ denote the space of complex valued measurable functions which are integrable with respect to Haar measure on T^j. If j is finite we shall write n instead of j. The subspaces $H^p(T^n)$ of $L^p(T^n)$, i.e. the Hardy spaces of T^n, have many well-known properties. A family of subspaces $H^p(T^\omega)$ of the $L^p(T^\omega)$ is defined and they are shown to have many of the same properties as the $H^p(T^n)$. However a major difference between $H^p(T^\omega)$ and $H^p(T^n)$ is observed. If $1 < p < \infty$ then $H^p(T^n)$ is complemented in $L^p(T^n)$, but $H^p(T^\omega)$ is uncomplemented in $L^p(T^\omega)$ for $1 < p < \infty$ unless $p = 2$.

Special properties of homogeneous functions in $H^1(T^\omega)$. Let j be a positive integer or ω. If j is finite we shall write n in place of j. We shall let T^n denote the compact group which is the Cartesian product of n circles, and T^ω the compact group which is the Cartesian product of countably many circles. The dual of T^n is the direct sum of n copies of the integers, and the dual of T^ω is the direct sum of countably many copies of the integers. If $g \in T^n$, then we write

$$g = (z_1, z_2, \cdots, z_n)$$

where each z_i is a complex number of unit modulus. If $g \in T^\omega$ it has a similar representation, but we must take a countable family, i.e.

$$g = (z_1, z_2, z_3, \cdots) .$$

By abuse of notation if $i \leqq n \leqq \infty$, we let z_i denote that $g \in T^n$ or $g \in T^\omega$ which has the following representation:

$$g = (1, \cdots, 1, z_i, 1, \cdots)$$

where z_i occurs in the ith place. We shall write m_n for the normalized Haar measure on T^n and m for the normalized Haar measure on T^ω.

The dual of T^n can be written as $\sum_{i=1}^n Z$, and if $x \in \sum_{i=1}^n Z$ then we write

$$x = (x_1, x_2, \cdots, x_n)$$

where each x_i is an integer. The dual of T^ω can be written as $\sum_{i=1}^\infty Z$, and if $x \in \sum_{i=1}^\infty Z$, then we write

$$x = (x_1, x_2, x_3, \cdots)$$

where each x_i is an integer, and for any particular x, only finitely many x_i are nonzero.

We define $A_n \subset \sum_{i=1}^{n} Z$ and $A \subset \sum_{i=1}^{\infty} Z$ by

$$A_n = \{x: x_i \geqq 0 \text{ for all } i\}$$
$$A = \{x: x_i \geqq 0 \text{ for all } i\} .$$

We need the following definitions to define $H^p(T^j)$. Although the definitions could be stated in terms of T^j it is easier to state them in the context of arbitrary compact abelian groups.

DEFINITION 1.1. Suppose G is a compact abelian group with dual group Γ. If $1 \leqq p \leqq \infty$ let $L^p(G)$ denote the space of complex valued measurable functions which are p^{th} power integrable with respect to Haar measure on G. If E is a subset of Γ, f will be called an *E-function* if $f \in L^1(G)$ and $\hat{f}(\gamma) = 0$ if $\gamma \in \Gamma \sim E$, where $\hat{f}(\gamma)$ is the Fourier transform of f evaluated at γ.

DEFINITION 1.2. Suppose $1 \leqq p \leqq \infty$ then $L_E^p(G) = \{f: f \in L^p(G)$ and f is an E-function$\}$.

DEFINITION 1.3.

$$H^p(T^n) = L_{A_n}^p(T^n)$$
$$H^p(T^\omega) = L_A^p(T^\omega) .$$

The properties of $H^p(T^n)$ are discussed in [7]. These spaces are related to analytic functions in several complex variables which are defined on the interior of the n-polydisc in C^n, and are subject to certain growth conditions near the distinguished boundary T^n. If $j = \omega$, there is no analogue of the interior of the n-polydisc. However we still have many of the nice properties of $H^p(T^n)$.

It is possible to imbed $H^p(T^n)$ in $H^p(T^\omega)$ in a natural way. We have the following homomorphisms

$$\begin{aligned} \pi_n : T^\omega &\longrightarrow T^n \\ (z_1, z_2, \cdots, z_n, z_{n+1} \cdots) &\longmapsto (z_1, z_2, \cdots z_n) \end{aligned}$$

and π_n induces an isometry I_n.

$$\begin{aligned} I_n : H^p(T^n) &\longrightarrow H^p(T^\omega) \\ f &\longmapsto f \circ \pi_n . \end{aligned} \tag{1}$$

DEFINITION 1.4. Suppose $f \in H^1(T^n)$ and s is a positive integer or

0. Then the s *homogeneous component of* $f = {}_nP_s(f)$, where ${}_nP_s(f)$ is defined by its Fourier transform

$$\widehat{{}_nP_s(f)}(x) = \begin{Bmatrix} \hat{f}(x) & \text{if } \sum x_i = s \\ 0 & \text{otherwise} \end{Bmatrix} .$$

That is if f has Fourier series

$$f(g) \sim \sum_{x \in A_n} a_x(g, x) ,$$

then ${}_nP_s(f)$ has the following Fourier series:

$${}_nP_s(f)(g) \sim \sum_{\substack{x \in A_n \\ \Sigma x_i = s}} a_x(g, x) .$$

Then ${}_nP_s(f)$ is a trigonometric polynomial since $\widehat{{}_nP_s(f)}$ has finite support.

DEFINITION 1.5. Suppose $f \in H^1(T^\omega)$ and $f = {}_nP_s(f)$ for some s. Then we say *f is homogeneous of degree s*. The previous definition is motivated by the following fact: If λ is a complex number of unit modulus and we write λ to mean the point $(\lambda, \lambda, \lambda, \cdots, \lambda)$ of T^n, then

$$f(\lambda g) = \lambda^s f(g) \quad \text{for all} \quad g \in T^n$$

if f is homogeneous of degree s. Clearly if f is homogeneous of degree s its Fourier transform has finite support, so f is a trigonometric polynomial and hence $f \in H^p(T^\omega)$ for $1 \leqq p \leqq \infty$. It is easy to show that ${}_nP_s$ is a bounded linear operator from $H^1(T^n)$ into $H^p(T^n)$ for each p. However it is not obvious that we can define an operator P_s on $H^1(T^\omega)$ which is analogous to ${}_nP_s$ on $H^1(T^n)$ because the sum that should define $P_s(f)$ for $f \in H^1(T^\omega)$ is not necessarily finite. The following lemma helps show that P_s can be defined as a bounded linear operator from $H^1(T^\omega)$ into $H^p(T^\omega)$.

LEMMA 1.6. *Suppose s is a positive integer or* 0, *and* $1 \leqq p \leqq \infty$. *Then there exists a projection P_s on $H^p(T^\omega)$ with $\|P_s\| = 1$ satisfying*:

$$\widehat{P_s f(x)} = \begin{Bmatrix} \hat{f}(x) & \textit{if } \Sigma x_i = s \\ 0 & \textit{otherwise} \end{Bmatrix} , \quad f \in H^p(T^\omega) .$$

That is if f has Fourier series

$$f(g) \sim \sum_{x \in A} a_x(g, x) ,$$

then $P_s(f)$ has the following Fourier series:

$$P_s(f)(g) \sim \sum_{\substack{x \in A \\ \Sigma x_i = s}} a_x(g, x) .$$

Proof. Consider the following subgroup H of $\sum_{i=1}^{\infty} Z$:

$$H = \left\{x : x \in \sum_{i=1}^{\infty} Z \quad \text{and} \quad \Sigma x_i = 0\right\}.$$

But $(\sum_{i=1}^{\infty} Z)/H$ is a quotient group of $\sum_{i=1}^{\infty} Z$ and hence its dual which we shall call D, is a compact subgroup of T^{ω}. Let m_D be normalized Haar measure on D. Since $D \subset T^{\omega}$, we can calculate the Fourier coefficients of m_D with respect to $\sum_{i=1}^{\infty} Z$. It is easy to calculate that

$$\hat{m}_D(x) = \chi_H(x) \quad \text{for all} \quad x \in \sum_{i=1}^{\infty} Z\,,$$

where $\chi_H(x)$ is the characteristic function of the set H. If s is a positive integer or 0, choose a $y_s \in \sum_{i=1}^{\infty} Z$ so that $\sum_{i=1}^{\infty} (y_s)_i = s$; then for the measure $y_s(g)dm_D(g)$

$$\widehat{y_s m_D}(x) = \hat{m}_D(x - y_s) = \left\{\begin{matrix} 1 & \text{if} \quad \Sigma(x - y_s) = 0 \\ & \text{i.e. } \Sigma(x)_i = s \\ 0 & \text{otherwise} \end{matrix}\right\}.$$

Evidently for all s

$$\int_G |y_s(g)\, dm_D(g)| = 1\,,$$

so if $f \in H^p(T^{\omega})$ we can consider $f * (y_s dm_D)$ where $*$ denotes the usual convolution of a measure on T^{ω} with a function which is in $H^p(T^{\omega})$, hence in $L^1(T^{\omega})$. We have the following inequalities:

$$(2) \qquad \| f * (y_s dm_D) \|_p \leqq \| f \|_p \int_G |y_s(g) dm_D(g)| = \| f \|_p\,.$$

If we calculate the Fourier transform of $f * (y_s dm_D)$

$$\widehat{f * (y_s dm_D)}(x) = \hat{f}(x)\widehat{(y_s dm_D)}(x) = \widehat{P_s(f)}(x)\,.$$

Since $f * (y_s dm_D)$ and $P_s(f)$ have the same Fourier transform they are the same element of $H^p(T^{\omega})$, and so from equation (2)

$$\| P_s(f) \|_p = \| f * (y_s dm_D) \|_p \leqq \| f \|_p$$

and this completes the proof.

DEFINITION 1.7. If $f \in H^p(T^{\omega})$, then the *s homogeneous component of f is $P_s(f)$.*

If $f = P_s(f)$ for some s, we say *f is homogeneous of degree s.* This definition is justified by the fact that if f is a homogeneous trigonometric polynomial of degree s on T^{ω}, then we have

$$(3) \qquad f(\lambda g) = \lambda^s f(g) \quad \text{for all} \quad g \in T^{\omega}$$

whenever λ is a complex number of unit modulus and on the left we write λ to mean $(\lambda, \lambda, \cdots)$.

Suppose that f is a homogeneous function and that $f \in H^1(T^j)$, where j is a positive integer or ω. If j is finite, then f is necessarily a trigonometric polynomial and the following lemma and theorem are obvious. However if $j = \omega$, f isn't necessarily a trigonometric polynomial, and the following lemma and theorem require proof.

LEMMA 1.8. *Suppose $f \in H^1(T^\omega)$ and that f is homogeneous of degree s. Then equation* (3) *is satisfied for almost all $g \in T^\omega$ and almost all λ.*

Proof. If f is a trigonometric polynomial there is nothing to prove. Otherwise by using an approximate identity we can find a sequence $\{f_n\}_{n=1}^\infty$ of homogeneous polynomials all of degree s such that

$$\lim_{n\to\infty} f_n = f$$

in the norm of $H^1(T^\omega)$. There exists a subsequence of $\{f_n\}_{n=1}^\infty$ say $\{f_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} f_{n_j}(g) = f(g) \text{ a.e.}$$

where a.e. means for almost all $g \in T^\omega$ with respect to Haar measure on T^ω. $T^\omega \times T$ is the product of the measure spaces T^ω and T, and so $T^\omega \times T$ is a measure space with the product measure.

Let

$$W = \{(g, \lambda) \in T^\omega \times T \text{ such that } f(\lambda g) = \lambda^s f(g)\} .$$

Then W is measurable and we wish to show that the measure of W is 1. Now consider any fixed $\lambda \in T$; we have

$$\lim_{j\to\infty} f_{n_j}(g) = f(g)$$
$$\lim_{j\to\infty} f_{n_j}(\lambda g) = f(\lambda g)$$

except for a null set of g. But for each j

$$f_{n_j}(\lambda g) = \lambda^s f_{n_j}(g) ,$$

$$f(\lambda g) = \lim_{j\to\infty} f_{n_j}(\lambda g) = \lim_{j\to\infty} \lambda^s f_{n_j}(g) = \lambda^s f(g)$$

except for a null set of g. So $m(W) = 1$, which finishes the proof.

The next theorem is an application of a theorem about $\Lambda(p)$ sets. We digress for a moment to define $\Lambda(p)$ set.

DEFINITION 1.9. Let G be a compact abelian group with dual group Γ. If $p > 1$ and $E \subset \Gamma$ we say E is a $\Lambda(p)$ set if $L^1_E(G) = L^p_E(G)$.

DEFINITION 1.10. If A is a subset of Γ and n is a positive integer we define $A^n = \{x \in \Gamma; x = a_1 + a_2 + \cdots + a_n$, where $a_i \in A$, $1 \leqq i \leqq n\}$.

THEOREM 1.11. *Suppose G is a compact abelian group with torsion-free dual group Γ. If E is an independent set in Γ, then E^s is a $\Lambda(p)$ set for all $p < \infty$ and all positive integers s.*

Proof. See [3, p. 28, Theorem 4].

THEOREM 1.12. *Suppose $f \in H^1(T^\omega)$ and that f is a homogeneous function of degree s where s is a positive integer or 0. Then $f \in H^p(T^\omega)$ for $1 \leqq p < \infty$.*

Proof. Let $E = \{z_i\}_{i=1}^\infty$. Then E is independent as a set in $\sum_{i=1}^\infty Z$ and so E^s is a $\Lambda(p)$ set for all $p < \infty$, by Theorem 1.11. But since $f \in H^1(T^\omega)$ and f is homogeneous of degree s, f is an E^s-function. By applying Theorem 1.11 we obtain that $f \in H^p(T^\omega)$ for all $p < \infty$, and this completes the proof.

COROLLARY 1.13. *Suppose $f \in H^1(T^\omega)$ and that f is a finite sum of homogeneous functions; then $f \in H^p(T^\omega)$ for $1 \leqq p < \infty$.*

Proof. By assumption f is a finite sum of homogeneous functions so we may write

$$f = \sum_{s=0}^{k} P_s(f) .$$

Since $f \in H^1(T^\omega)$ each $P_s(f) \in H^1(T^\omega)$ for $0 \leqq s \leqq k$. By Theorem 1.12 each $p_s(f) \in H^p(T^\omega)$ for $1 \leqq p < \infty$, so f is a finite sum of functions in $H^p(T^\omega)$ hence $f \in H^p(T^\omega)$.

Theorem 1.12 is really a theorem about $H^1(T^\omega)$ rather than $L^1(T^\omega)$. In that context Theorem 1.12 is false. In fact Theorem 1.12 is false even for $L^1(T^2)$ and hence for $L^1(T^\omega)$.

If j is a positive integer or ∞, we define homogeneity for arbitrary functions in $L^1(T^j)$ as follows: If $f \in L^1(T^j)$, we say f is homogeneous of degree s if

$$\hat{f}(x) = 0 \text{ if } x \in \sum_{i=1}^{j} Z \text{ and } \Sigma x_i \neq s .$$

To show that Theorem 1.12 can't be extended to $L^1(T^2)$, we shall construct for every $p > 1$ and for every positive integer N, a homogeneous polynomial f of degree 0 on T^2 such that

$$||f||_1 = 1$$
$$||f||_p \geqq N .$$

For given $p > 1$, find a trigonometric polynomial b defined on T such that

$$||b||_1 = 1$$
$$||b||_p \geqq N$$

where $b(z_1)$ has Fourier series

$$b(z_1) = \sum_{k=0}^{t} a_k z_1^k .$$

Define the polynomial f by

$$f(z_1, z_2) = \sum_{k=0}^{t} a_k z_1^k z_2^{-k} .$$

We wish to compute the norm of f in $L^1(T^2)$ and in $L^p(T^2)$:

$$\begin{aligned} ||f||_1 &= \int_{T^2} |f(z_1, z_2)|\, dm_1(z_1) dm_2(z_2) \\ &= \int_{T^2} \left| \sum_{k=0}^{t} a_k (z_1 z_2^{-1})^k \right| dm_1(z_1) dm_2(z_2) \\ &= \int_{T^2} \left| \sum_{k=0}^{t} a_k (z_1)^k \right| dm_1(z_1) dm_2(z_2) = \int_T ||b||_1 dm_2(z_2) = \int_T 1\, dm_2(z_2) = 1 . \end{aligned}$$

The crucial equality in equation (4) is justified by the translation invariance of $dm_1(z_1)$. By a similar computation we have

$$||f||_p = ||b||_p \geqq N$$

and this provides the desired counterexample.

2. A convergence theorem for $H^p(T^\omega)$. By the M. Riesz theorem on conjugate functions [8], if $1 < p < \infty$ and $f \in H^p(T)$, then

$$f = \lim_{n\to\infty} \sum_{s=0}^{n} a_s z_1^s , \qquad a_s = \hat{f}(s)$$

in the norm of $H^p(T)$. In our terminology this can be written

$$f = \lim_{n\to\infty} \sum_{s=0}^{n} {}_1P_s(f) .$$

The next theorem gives an analogous result for $H^p(T^\omega)$. The proof uses a theorem about ordered groups so we digress for a moment to define the relevant terms.

Suppose Γ is a discrete abelian group and P is a subset of Γ with the following properties:

1. If $\gamma_1 \in P$ and $\gamma_2 \in P$ then $\gamma_1 + \gamma_2 \in P$.

If $-P$ denotes the set whose elements are the inverses of the elements of P then we have

2. $P \cap (-P) = \{0\}$
3. $P \cup (-P) = \Gamma$.

Under these conditions P induces an order in Γ as follows: For γ_1 and γ_2 elements of Γ, say $\gamma_1 \geqq \gamma_2$ if $\gamma_1 - \gamma_2 \in P$. It is easy to check that this is a linear order. A given group may have many different orders corresponding to different choices of P with the three properties above.

DEFINITION 2.1. Suppose G is a compact abelian group whose dual group Γ is ordered. Let f be a trigonometric polynomial on G with Fourier series

$$f(g) \sim \sum_{\gamma \in \Gamma} a_\gamma(g, \gamma) .$$

Define $\Phi(f)$ by

$$\Phi(f)(g) \sim \sum_{\substack{\gamma \in \Gamma \\ \gamma \geqq 0}} a_\gamma(g, \gamma) .$$

We shall need the following generalization of the M. Riesz theorem on conjugate functions. It is due to Bochner [1].

THEOREM 2.2. *Suppose $1 < p < \infty$. Then there exists a constant A_p, independent of G or the particular order in Γ such that if f is a trigonometric polynomial on G, then*

$$|| \Phi(f) ||_p \leqq A_p || f ||_p .$$

THEOREM 2.3. *Let $1 < p < \infty$. Then if $f \in H^p(T^\omega)$*

$$\lim_{n \to \infty} \sum_{s=0}^{n} P_s(f) = f$$

in the norm of $H^p(T^\omega)$.

Proof. Fix p. Define Y_n by

$$Y_n(f) = \sum_{s=0}^{n} P_s(f) \text{ if } f \in H^p(T^\omega) .$$

Clearly trigonometric polynomials are dense in $H^p(T^\omega)$ and

$$\lim_{n \to \infty} Y_n(f) = f$$

whenever f is a trigonometric polynomial. It remains to show that the family $\{Y_n\}_{n=1}^{\infty}$ is uniformly bounded on trigonometric polynomials, i.e.

$$\| Y_n(f) \|_p \leqq K \| f \|_p$$

f a trigonometric polynomial where K is a positive constant independent of n and f. Then by a standard argument in functional analysis, the proof is complete. I shall show that the norm of Y_n is majorized by A_p, where A_p is the constant of Theorem 2.2.

Our first task is to induce an order in $\sum_{i=1}^{\infty} Z$ so that we can apply Theorem 2.2. First choose a family $\{d_i\}_{i=1}^{\infty}$ of real numbers which satisfies the following properties:

1. $d_1 = -1$, $-1 < d_i < -n/(n+1)$ for $i \neq 1$.
2. The set $\{d_i\}$ is independent in the group sense as a subset of the reals.

We define a homomorphism from $\sum_{i=1}^{\infty} Z$ into the reals by

$$\pi \colon \sum_{i=1}^{\infty} \longrightarrow R$$

$$x \longmapsto \sum_{i=1}^{\infty} d_i x_i \,.$$

π is clearly a homomorphism; since the d_i are linearly independent, it has a trivial kernel, i.e. if $\pi(x) = 0$ then $x = 0$. Define

$$P = \left\{ x \colon x \in \sum_{i=1}^{\infty} Z \text{ and } \pi(x) \geqq 0 \right\} .$$

Then P satisfies the necessary properties to induce an order in $\sum_{i=1}^{\infty} Z$. If $f(g)$ is an arbitrary trigonometric polynomial on T^{ω} define a trigonometric polynomial $f_1(g)$ as follows:

$$f_1(g) = z_1^{-n}(g) f(g) \,.$$

Let $f(g) = \Sigma a_x(g, x)$. Then

$$f_1(g) = z_1^{-n}(g) f(g) = \Sigma a_x(g, -nz_1)(g, x) = \Sigma a_x(g, x - nz_1)$$

and

$$\phi(f_1) = \sum_{\pi(x - nz_1) \geqq 0} a_x(g, x - nz_1) \,.$$

If $\pi(x - nz_1) \geqq 0$, then

$$0 \leqq \pi(x - nz_1) = \pi(x) + \pi(-nz_1) = \pi(x) - n\pi(z_1) = \pi(x) + n$$

and $\pi(x) \geqq -n$. But $\pi(x) = \Sigma d_i x_i$, and by using property 1 of $\{d_i\}$ it is clear that $\pi(x) \geqq -n$ if and only if $\Sigma x_i \leqq n$. So $\phi(f_1) = \Sigma a_x(g, x - nz_1)$.

Then it is easy to compute that $\Sigma x_i \leqq n$

$$z_1^n \Phi(f_1) = \sum_{i=1}^{n} P_i(f) = Y_n(f) .$$

By Theorem 2.2 we have that

$$\|\Phi(f_1)\|_p \leqq A_p \|f_1\|_p .$$

So we have

$$\begin{aligned} &\|Y_n(f)\|_p = \|z_1^n \Phi(f_1)\|_p = \|\Phi f_1\|_p \leqq A_p \|f_1\|_p \\ &= A_p \|z_1^{-n} f\|_p = A_p \|f\|_p , \end{aligned}$$

so the norm of Y_n is less than or equal to A_p and the proof is complete.

3. The complementation problem.

The next theorem shows that $H^p(T^\omega)$ is uncomplemented as a subspace of $L^p(T^\omega)$ if $p \neq 2$. This is in contrast to $H^p(T^n)$ which is complemented in $L^p(T^n)$ except when $p = 1$ or $p = \infty$. Although other examples of uncomplemented subspaces of an L^p space are known, $H^p(T^\omega)$ has the advantage of being defined in a concrete way.

DEFINITION 3.1. Let G be a compact abelian group. If $f \in L^1(G)$ let f_{g_0} denote the *g_0-translate of f* where

$$f_{g_0}(g) = f(g_0 + g) .$$

LEMMA 3.2. *Let G be a compact abelian group with dual group Γ. Suppose $1 \leqq p < \infty$ and that T is a bounded projection from $L^p(G)$ onto $L^p_E(G)$. Then a linear operator Q can be defined by*

$$Q(f) = \int_G [T(f_g)]_{-g} dm(g) \qquad f \in L^p(G) ,$$

where the integral is the Bochner integral.

Q is the natural projection from $L^p(G)$ onto $L^p_E(G)$, i.e., if $f \in L^p(G)$ then $Q(f)$ is defined by its Fourier transform as follows:

$$\widehat{G(f)}(x) = \begin{Bmatrix} \hat{f}(x) & x \in E \\ 0 & \text{otherwise} \end{Bmatrix} .$$

Proof. The proof for the case $G = T, \Gamma = Z, E = Z^+, p = 1$ is given [4, page 154]. The proof in the general case is analogous.

THEOREM 3.3. *Suppose $p \neq 2$, then $H^p(T^\omega)$ is uncomplemented as subspace of $L^p(T^\omega)$.*

Proof. If $p = 1$ or $p = \infty$, there is really nothing to prove. There is a theorem in [4, pp. 154–155] which proves that $H^1(T)$ is uncomplemented in $L^1(T)$, and that $H^\infty(T)$ is uncomplemented in $L^\infty(T)$. Then since $H^i(T)$ and $L^i(T)$ can be isometrically embedded into $H^i(T^\omega)$ and $L^i(T^\omega)$ respectively for $i = 1, \infty$, the theorem is proved for $p = 1$ or $p = \infty$. In any case the argument which follows is valid for $p = 1$, and with slight modifications for $p = \infty$.

Let S be the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is defined on trigonometric polynomials by

$$\begin{aligned} S\colon L^p(T^\omega) &\longrightarrow H^p(T^\omega) \\ f \quad &\longmapsto \quad S(f) \end{aligned}$$

where

$$\widehat{S(f)}(x) = \begin{Bmatrix} \hat{f}(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{Bmatrix} .$$

We wish to show that S can't be extended to a bounded operator defined on all of $L^p(T^\omega)$. To do this it is sufficient to find trigonometric polynomials f_n on T^ω such that

$$||f_n||_p = 1 \tag{5}$$

$$||S(f_n)||_p = (1 + \varepsilon)^n \quad \text{where} \quad \varepsilon > 0 . \tag{6}$$

By [8, p. 295, Ex. 2] we can find a trigonometric polynomial h defined on T so that

$$h(z_1) = \sum_{k=-n}^{n} a_k z_1^k \qquad ||h||_p = 1$$

and if

$$h_+(z_1) = \sum_{k=0}^{n} a_k z_1^k$$

then we have

$$||h_+||_p = 1 + \varepsilon$$

where ε is some positive number which depends upon p. Consider the trigonometric polynomial r defined on T^2 by

$$r(z_1, z_2) = h(z_1)h(z_2) = \Big(\sum_{k=-n}^{n} a_k z_1^k\Big)\Big(\sum_{k=-n}^{n} a_k z_2^k\Big) .$$

Define r_+ by

$$r_+(z_1, z_2) = h_+(z_1)h_+(z_2) = \Big(\sum_{k=0}^{n} a_k z_1^k\Big)\Big(\sum_{k=0}^{n} a_k z_2^k\Big) .$$

Then it is easy to compute that

$$\|r\|_p = \|h\|_p^2 = 1$$
$$\|r_+\|_p = (\|h_+\|_p)^2 = (1+\varepsilon)^2 \,.$$

We define trigonometric polynomials on T^ω by

$$f_1 = I_1(h) \qquad f_2 = I_2(r)$$

where I_1 and I_2 were defined in equation (1). It is easy to check that

$$S(f_1) = I_1(h_+) \qquad S(f_2) = I_2(r_+)$$

and since I_1 and I_2 are isometries we have

$$\|f_1\|_p = \|I_1(h)\|_p = \|h\|_p = 1$$
$$\|S(f_1)\|_p = \|I_1(h_+)\|_p = \|h_+\|_p = 1+\varepsilon$$
$$\|f_2\|_p = \|I_2(r)\|_p = \|r\|_p = 1$$
$$\|S(f_2)\|_p = \|I_2(r_+)\|_p = \|r_+\|_p = (1+\varepsilon)^2 \,.$$

By a similar argument we can construct trigonometric polynomials $f_3, f_4, \cdots$ and hence f_n for any n and f_n will satisfy equations (5) and (6). This shows that the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$ isn't bounded. To finish the proof we must show there is no bounded projection of any kind from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is the identity when restricted to $H^p(T^\omega)$.
Suppose there exists $\tilde{S}$ a linear transformation from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is the identity when restricted to $H^p(T^\omega)$. Define a linear operator Q by

$$Q(f) = \int_{T^\omega} [\tilde{S}(f_g)]_{-g}\, dm(g)$$

where the integral is the Bochner integral. Then Q is a bounded linear operator from $L^p(T^\omega)$ into $H^p(T^\omega)$ and by Lemma 3.2 we have that $Q = S$, where S is the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$. But we know that S isn't a bounded projection and this provides the contradiction which finishes the proof.

We wish to thank Professor Henry Helson for his aid and encouragement in writing our thesis. We would also like to thank Professor Alessandro Figà-Talamanaca for many helpful discussions.

References

1. S. Bochner, *Additive set functions on groups*, Ann. Math., **40** (1939), 769-799.
2. A. Bonami, *Ensembles $\Lambda(p)$ dans le dual de D^∞*, Ann. Inst. Fourier Grenoble **18** (1968), 193-204.

3. ———, Thesis, Universite De Paris, Faculte Des Sciences D'Orsay, 1970.
4. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
5. W. Rudin, *Trigonometric series with gaps*, J. Math. and Mech., **9** (1960), 203-228.
6. ———, *Fourier Analysis on Groups*, Interscience Publishers, New York, 1960.
7. ———, *Function Theory in Polydiscs*, W. A. Benjamin, New York, 1969.
8. A. Zygmund, *Trigonometric Series*, Cambridge University Press, 1959.

Received April 12, 1971 and in revised form August 13, 1971. This paper constitutes a section of the author's doctoral dissertation submitted to the University of California in June 1970, under the title, *Harmonic Analysis on Compact Abelian Groups*. The material in Section 3 was presented to the American Mathematical Society on January 22, 1971.

UNIVERSITY OF CALIFORNIA, BERKELEY
AND
WAYNE STATE UNIVERSITY