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A CLASS OF OPERATORS ON EXCESSIVE FUNCTIONS

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Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a special standard Markov process with state space $(E, \mathscr{E})$ and transition semigroup (P_t). We emphasize here that the $\mathscr{F}_t$ are the usual completions of the natural σ-fields for the process. In this paper, we associate with certain multiplicative functionals of X operators on the class of excessive functions which are related to the operators P_M but which are a bit unusual in probabilistic potential theory in that they are not generally determined by kernels on $E \times \mathscr{E}$. An application is given to a problem treated by P.-A. Meyer concerning natural potentials dominated by an excessive function.

2. The operator associated with a natural multiplicative functional.[1] By a multiplicative functional of X, we mean a progressively measurable process M which satisfies, in addition to the standard conditions ([1], III, (1.1)) the following condition:

(2.1) almost surely, $M_\zeta = 0$, $t \to M_t$ is decreasing on $[0, \infty)$ and if $S = \inf\{t > 0: M_t = 0\}$, then $t \to M_t$ is right continuous on $[0, S)$, and $M_t M_S \circ \theta_t = M_{t+S\circ\theta_t}$ a.s. for all $t \geqq 0$.

A simple example which illustrates some possibilities is obtained by considering X to be uniform motion to the right on the real line and $M_t = f(X_t)/f(X_0)$ on $\{f(X_0) > 0\}$, $M_t = 0$ for all t on $\{f(X_0) = 0\}$, where f is a decreasing positive function on the line, $f(0+) = 0$, f is right continuous on $(-\infty, 0)$ and $f(0) \leqq f(0-)$.

If M is a multiplicative functional, then S is a terminal time and so $M_t 1_{[0,S)}(t)$ is a multiplicative functional which is right continuous. For a given M, the modified functional will be denoted $\tilde{M}$. Let us denote by E_M the set $\{x \in E: P^x\{S > 0\} = 1\} = E_{\tilde{M}}$ and call M exact if $\tilde{M}$ is exact. Note that M and $\tilde{M}$ generate the same resolvent, but not necessarily the same semigroup.

It should be emphasized that one will not have the freedom to replace M by an equivalent multiplicative functional, for the operator to be associated with M will not respect equivalence.

Let M be a given MF; for almost all ω, let $(-dM_t(\omega))$ denote the measure on $(0, \zeta(\omega))$ generated by the increasing function $t \to 1 - M_{t\wedge S}(\omega)$. Care should be taken when computing with $(-dM_t)$, since $(-dM_t)$ is generally not the restriction of $(-d\tilde{M}_t)$ to $(0, S]$.

[1] The reader is referred to the books of Blumenthal and Getoor [1] and Meyer [2] for unexplained terminology.

DEFINITION 2.2. A multiplicative functional M is called natural if, almost surely, the trajectories $t \to M_t$ and $t \to X_t$ have no common discontinuity on $[0, S)$, and $X_S = X_{S-}$ on $\{M_S < M_{S-}, S < \zeta\}$.

We now associate with a natural MF M an operator $\bar{P}_M^\alpha$ on the class $\mathscr{S}^\alpha$ of α-excessive functions for X.

DEFINITION 2.3. If M is a natural MF and $f \in S^\alpha$, let

$$\begin{aligned}\bar{P}_M^\alpha f(x) &= E^x\left\{\int_{(0,\zeta)} e^{-\alpha t} f(X_t)_-(-dM_t) + e^{-\alpha S} f(X_S) M_S\right\}, \quad x \in E_M \\ &= f(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \quad x \notin E_M .\end{aligned}$$

By $f(X_t)_-$ is meant the left limit of the trajectory $s \to f(X_s)$ at t if $t > 0$, and $f(X_0)$ if $t = 0$. Recall that if M is a right continuous MF, $\alpha \geqq 0$ and $\mathscr{E}_+^*$, one defines $P_M^\alpha f$ by

$$\begin{aligned}P_M^\alpha f(x) &= E^x \int_{(0,\zeta)} e^{-\alpha t} f(X_t)(-dM_t) , \quad x \in E_M \\ &= f(x) \qquad\qquad\qquad\qquad\qquad , \quad x \notin E_M .\end{aligned} \tag{2.4}$$

One obtains $P_M^\alpha U^\alpha f + V^\alpha f = U^\alpha f$, where (V^α) is the resolvent for the subprocess (X, M) and it follows that if M is exact, $P_M^\alpha g \in \mathscr{S}^\alpha$ for all $g \in \mathscr{S}^\alpha$. If $f \in \mathscr{S}^\alpha$ is regular, in particular if $f = U^\alpha g$ for some $g \in \mathscr{E}_+^*$, then for M natural, $\bar{P}_M^\alpha f = P_{\underset{\sim}{M}}^\alpha f$. In general though, the trajectory $t \to f(X_t)$ can jump at the same time as does the trajectory $t \to M_t$ and $\bar{P}_M^\alpha f$ will differ from $P_{\underset{\sim}{M}}^\alpha f$. Because of the assumption that X is special standard, it follows from [1], IV, (4.21) that $f(T_T)_- \geqq f(X_T)$ for any accessible stopping time T, and therefore

$$\bar{P}_M^\alpha f(x) \geqq P_{\underset{\sim}{M}}^\alpha f(x) \text{ for all } x \text{ if } f \in \mathscr{S}^\alpha . \tag{2.5}$$

We shall show that $\bar{P}_M^\alpha f \leqq f$ and $\bar{P}_M^\alpha f \in \mathscr{S}^\alpha$ if $f \in \mathscr{S}^\alpha$. The fact that the action of $\bar{P}_M^\alpha$ on α-potentials is the same as that of $P_{\underset{\sim}{M}}^\alpha$, but that $\bar{P}_M^\alpha$ may differ from $P_{\underset{\sim}{M}}^\alpha f$ shows that generally, $\bar{P}_M^\alpha$ is not determined by a kernel on $E \times \mathscr{E}$.

The first lemma shows that although it may not be determined by a kernel, $\bar{P}_M^\alpha$ does respect certain increasing limits. Obviously $\bar{P}_M^\alpha f \leqq \bar{P}_M^\alpha g$ if $f, g \in \mathscr{S}^\alpha$ and $f \leqq g$.

LEMMA 2.6. *If $f \in \mathscr{S}^\alpha$, $\bar{P}_M^\alpha(f \wedge n)$ increases to $\bar{P}_M^\alpha f$ as $n \to \infty$.*

Proof. It suffices to prove that $(f \wedge n)(X_t)_-$ increases to $f(X_t)_-$ for all $t \in (0, \zeta)$, almost surely. If the trajectory $s \to f(X_s)$ is right continuous and has left limits on $(0, \zeta)$, then for each $t < \zeta$, if $f(X_t)_- > \beta$, then there exists $\varepsilon > 0$ such that $f(X_s) > \beta$ on $[t - \varepsilon, t)$. Therefore, if $n > \beta$, $(f \wedge n)(X_s) > \beta$ on $[t - \varepsilon, t)$ and hence $(f \wedge n)(X_t)_- \geqq \beta$.

We remark at this point that $\alpha \to \bar{P}_M^\alpha f(x)$ is right continuous for every fixed choice of M, f and x.

THEOREM 2.7. *If M is an exact natural MF, $0 \leqq \alpha < \infty$ and $f \in \mathscr{S}^\alpha$, then $\bar{P}_M^\alpha f \leqq f$ and $\bar{P}_M^\alpha f \in \mathscr{S}^\alpha$.*

Proof. Because of (2.6) it may be assumed that f is bounded. We may also assume $\alpha > 0$, since the case $\alpha = 0$ will follow by a trivial limit argument. Let

$$\begin{aligned} N_t &= M_t,\ t < S \\ &= M_S,\ t \geqq S \text{ on } \{S < \zeta\} \\ &= M_{\zeta-},\ t \geqq \zeta \text{ on } \{S = \zeta\}\ . \end{aligned}$$

One then has $-dN_t = -dM_t$ almost surely, and for $x \in E_M$, $\bar{P}_M^\alpha f(x) = E^x\left\{\int_0^\infty e^{-\alpha t} f(X_t)_-(-dN_t) + e^{-\alpha S} f(X_S) M_S\right\}$. Define a family $\{T_s;\ 0 < s < 1\}$ of $(\mathscr{F}_t)$ stopping times by

$$T_s = \inf\{u > 0 : 1 - N_u > s\}\ .$$

It is clear that $s \to T_s$ is almost surely increasing and right continuous, $T_s = \infty$ a.s. on $\{T_s > S\}$, $\{T_s = 0 \text{ for some } s\} = \{M_{0+} = 0\}$ and $\{T_s \leqq S\} = \{T_s < \zeta\}$ almost surely. By the change of variable formula,

$$\int_{(0,\zeta)} e^{-\alpha t} f(X_t)_-(-dM_t) = \int_0^1 e^{-\alpha T_s} f(X_{T_s})_- 1_{\{T_s < \zeta\}} ds\ .$$

Let $Z_t = e^{-\alpha(t \wedge S)} f(X_{t \wedge S})$. Since $\alpha > 0$,

$$\begin{aligned} \int_0^1 Z_{T_s-} ds &= \int_0^1 Z_{T_s-} 1_{\{T_s \leqq S\}} ds + \int_0^1 Z_{T_s-} 1_{\{T_s = \infty\}} ds \\ &= \int_0^1 e^{-\alpha T_s} f(X_{T_s})_- 1_{\{T_s \leqq S\}} ds + \int_0^1 e^{-\alpha S} f(X_s) 1_{\{T_s = \infty\}} ds \\ &= \int_{(0,\zeta)} e^{-\alpha t} f(X_t)_-(-dM_t) + e^{-\alpha S} f(X_S) M_S\ . \end{aligned}$$

Upon checking separately the case $x \notin E_M$, one finds

$$\bar{P}_M^\alpha f(x) = E^x \int_0^1 Z_{T_s-} ds,\ x \in E\ . \tag{2.8}$$

We now need a fact which will be of use at a subsequent point in the proof.

(2.9) For any initial measure μ, the set of $s \in (0, 1)$ for which T_s is a.s. P^μ equal to an accessible stopping time has full Lebesgue measure.

To demonstrate (2.9), we let

$$I(\omega) = \{\infty\} \cup [0, \zeta(\omega)) - \{t \in (0, \zeta(\omega)): N_{t+\varepsilon}(\omega) < N_t(\omega) \text{ for all } \varepsilon > 0 \text{ and } N_{t-\varepsilon}(\omega) = N_t(\omega) \text{ for some } \varepsilon > 0\} .$$

Obviously $[0, \zeta) - I$ is countable and $\int_{[0,\zeta)-I}(-dM_t) = 0$ a.s., and consequently $\int_0^1 1_{\{T_s \notin I\}} ds = 0$ a.s., by the change of variable formula. If we prove that T_s is accessible on $\{T_s \in I\}$, we shall have proven (2.9), for by Fubini,

$$0 = E^\mu \int_0^1 1_{\{T_s \notin I\}} ds = \int_0^1 P^\mu\{T_s \notin I\} ds .$$

On $\{T_s = 0\} \cup \{T_s = \infty\}$, T_s is trivially accessible. It is easy to check that $\{T_s \in I, 0 < T_s < \zeta\} = \{0 < T_s = T_{s-} < \zeta\}$, and on $\{T_s \in I, 0 < T_s < \zeta\} \cap \{X_{T_s} = X_{T_s-}\}$, T_s is accessible by the famous theorem of Meyer, whilst on $\{T_s \in I, 0 < T_s < \zeta\} \cap \{X_{T_s} \neq X_{T_s-}\}$, $N_{T_s} = N_{T_s-}$ since M is natural, and it follows that a.s., $T_{s-\varepsilon} < T_s$ for all $\varepsilon \in (0, s)$. The accessibility of T_s on $\{T_s \in I\}$ is now evident.

To obtain $\bar{P}_M^\alpha f \leqq f$, we invoke (2.8) to see that $\bar{P}_M^\alpha f(x) = \int_0^1 E^x Z_{T_s-} ds$, and conclude by observing that $(Z_t, \mathscr{F}_t, P^x)$ is a bounded non-negative right-continuous supermartingale and that for almost all $s \in (0, 1)$, T_s is a.s. P^x accessible to find $E^x Z_{T_s-} \leqq E^x Z_0 = f(x)$ for almost all s.

We prove next that $\bar{P}_M^\alpha f$ is α-super-mean-valued. It is enough to give a proof in case $\alpha > 0$. From (2.8) we see that

$$P_t^\alpha \bar{P}_M^\alpha f(x) = E^x e^{-\alpha t} E^{X_t} \int_0^1 Z_{T_s-} ds = \int_0^1 E^x e^{-\alpha t} Z_{T_s-} \circ \theta_t ds .$$

Our first step is to show

$$P_t^\alpha \bar{P}_M^\alpha f(x) \leqq \int_0^1 E^x (Z_{t+T_s \circ \theta_t})_- ds , \qquad x \in E . \tag{2.10}$$

On $\{S \geqq t + T_s \circ \theta_t\}$, either $S > t$ or $S = t$ and $T_s \circ \theta_t = 0$. It is a matter of checking cases to see that

$$e^{-\alpha t} Z_{T_s-} \circ \theta_t = (Z_{t+T_s \circ \theta_t})_- \text{ on } \{S > t\} ,$$

and a.s. on $\{S = t, T_s \circ \theta_t = 0\}$,

$$e^{-\alpha t} Z_{T_s-} \circ \theta_t = e^{-\alpha t} f(X_t) = e^{-\alpha t} f(X_{t-}) \leqq e^{-\alpha t} f(X_t)_- = (Z_{t+T_s \circ \theta_t})_- .$$

Hence $e^{-\alpha t} Z_{T_s-} \circ \theta_t \leqq (Z_{t+T_s \circ \theta_t})_-$ a.s. on $\{S \geqq t + T_s \circ \theta_t\}$. On $\{S < t + T_s \circ \theta_t\}$, $(Z_{t+T_s \circ \theta_t})_- = e^{-\alpha S} f(X_S)$, while

$$\begin{aligned} e^{-\alpha t} Z_{T_s-} \circ \theta_t &\leqq e^{-\alpha(t+T_s \circ \theta_t)} f(X_{t+T_s \circ \theta_t})_- \text{ on } \{S < t + T_s \circ \theta_t, T_s \circ \theta_t \leqq S \circ \theta_t\} , \\ &= e^{-\alpha(t+S \circ \theta_t)} f(X_{t+S \circ \theta_t}) \text{ on } \{S < t + T_s \circ \theta_t, T_s \circ \theta_t > S \circ \theta_t\} . \end{aligned}$$

One sees readily from (2.9) that for fixed x, $t + T_s \circ \theta_t$ is a.s. P^x equal to an accessible stopping time for almost all s and so for almost all choices of s, there exists an increasing sequence $\{R_n\}$ of stopping times with limit $t + T_s \circ \theta_t$ such that $P^x\{R_n < t + T_s \circ \theta_t\} = 1$ for every n. Then $L_n = R_n \wedge (t + S \circ \theta_t)$ increases to $t + T_s \circ \theta_t$ strictly from below (a.s. P^x) on $\{S < t + T_s \circ \theta_t, T_s \circ \theta_t \leqq S \circ \theta_t\}$ and R_n is eventually equal to $t + S \circ \theta_t$ on $\{S < t + T_s \circ \theta_t, T_s \circ \theta_t > S \circ \theta_t\}$. One then has

$$
\begin{aligned}
E^x e^{-\alpha t} Z_{T_s-} \circ \theta_t &= E^x\{e^{-\alpha t} Z_{T_s-} \circ \theta_t [1_{\{S \geqq t+T_s\circ\theta_t\}} + 1_{\{S<t+T_s\circ\theta_t\}}]\} \\
&\leqq E^x\{(Z_{t+T_s\circ\theta_t})_- 1_{\{S\geqq t+T_s\circ\theta_t\}} + \lim_n e^{-\alpha L_n} f(X_{L_n}) 1_{\{S<t+T_s\circ\theta_t\}}\} .
\end{aligned}
$$

But $t + S \circ \theta_t \geqq S$ a.s. and so $L_n \geqq S$ eventually, a.s., on $\{S < t + T_s \circ \theta_t\}$ and it follows from the fact that $\{e^{-\alpha t} f(X_t), \mathscr{F}_t, P^x\}$ is a bounded nonnegative right-continuous supermartingale that $E^x e^{-\alpha t} Z_{T_s-} \circ \theta_t \leqq E^x(Z_{t+T_s\circ\theta_t})_-$ for almost all $s \in (0, 1)$. This proves (2.10).

Now observe that a.s., $T_s \leqq t + T_s \circ \theta_t$ on $\{T_s \leqq S\}$ and $t + T_s \circ \theta_t > S$ on $\{T_s > S\}$. For, on $\{T_s \leqq S\} \cap \{M_t > 0\}$,

$$
\begin{aligned}
t + T_s \circ \theta_t &= \inf\{u + t : u > 0, N_u \circ \theta_t < 1 - s\} \\
&\geqq \inf\{u + t : u > 0, M_u \circ \theta_t < 1 - s\} \\
&= \inf\{v > t : M_v < (1 - s) M_t\} \\
&\geqq \inf\{v > 0 : M_v < 1 - s\} = T_s ,
\end{aligned}
$$

and on $\{T_s \leqq S\} \cap \{M_t = 0\}$, $t \geqq S$ so $T_s \leqq S \leqq t \leqq t + T_s \circ \theta_t$. On $\{T_s > S\} \cap \{M_t > 0\}$, the same calculation as above gives $t + T_s \circ \theta_t \geqq \inf\{v > 0 : M_v < 1 - s\}$ a.s., and so $t + T_s \circ \theta_t \leqq S$ would imply $T_s \leqq S$. On $\{T_s > S\} \cap \{M_t = 0\}$, $M_S > 0$ so $t > S$ and $t + T_s \circ \theta_t > S$ almost surely.

For almost all $s \in (0, 1)$, T_s and $t + T_s \circ \theta_t$ are (a.s. P^x) accessible stopping times and it follows simply from the order relation observed above and the fact that $(Z_t, \mathscr{F}_t, P^x)$ is bounded nonnegative supermartingale that $E^x(Z_{t+T_s\circ\theta_t})_- \leqq E^x Z_{T_s-}$ for almost all $s \in (0, 1)$, whence $P_t^\alpha \bar{P}_M^\alpha f(x) \leqq \bar{P}_M^\alpha f(x)$.

It remains to show $P_t^\alpha \bar{P}_M^\alpha f(x) \to \bar{P}_M^\alpha f(x)$ as $t \to 0$. If $x \in E_M$, then $X_t \in E_M$ a.s. on $\{t < S\}$, and so

$$
\begin{aligned}
P_t^\alpha \bar{P}_M^\alpha f(x) &= E^x e^{-\alpha t} \bar{P}_M^\alpha f(X_t) \\
&\geqq E^x e^{-\alpha t} 1_{\{t<S\}} E^{X_t}\left\{\int_{(0,\zeta)} e^{-\alpha s} f(X_s)_- (-dM_s) + f(X_S) M_S e^{-\alpha S}\right\} \\
&= E^x 1_{\{t<S\}}\left\{\int_{(0,\zeta\circ\theta_t)} e^{-\alpha(t+s)} f(X_{t+s})_- (-dM_s \circ \theta_t) + f(X_{t+S\circ\theta_t}) M_S \circ \theta_t e^{-\alpha S\circ\theta_t}\right\} \\
&= E^x 1_{\{t<S\}} M_t^{-1}\left\{\int_{(t,\zeta)} e^{-\alpha u} f(X_u)_- (-dM_u) + f(X_S) M_S e^{-\alpha S}\right\} .
\end{aligned}
$$

By Fatou's lemma, if $x \in E_M$

$$\begin{aligned}
&\liminf_{(t\to 0)} P_t^\alpha \bar{P}_M^\alpha f(x) \\
&\geqq E^x \lim_{(t\to 0)} 1_{\{t<S\}} M_t^{-1}\left\{\int_{(t,\zeta)} e^{-\alpha u} f(X_u)_-(-dM_u) + f(X_S)M_S e^{-\alpha S}\right\} \\
&= E^x\left\{\int_{(0,\zeta)} e^{-\alpha u} f(X_u)_-(-dM_u) + e^{-\alpha S} M_S f(X_S)\right\} = \bar{P}_M^\alpha f(x) .
\end{aligned}$$

Consequently $P_t^\alpha \bar{P}_M^\alpha f(x) \to \bar{P}_M^\alpha f(x)$ if $x \in E_M$. On the other hand, if $x \in E - E_M$, $P_t^\alpha \bar{P}_M^\alpha f(x) \geqq P_t^\alpha P_{\tilde{M}}^\alpha f(x)$ which converges as $t \to 0$ to $P_{\tilde{M}}^\alpha f(x) = f(x) = \bar{P}_M^\alpha f(x)$, using exactness of $\tilde{M}$. Our proof is now complete.

3. Application to a problem treated by Meyer. Meyer [3] proved that if u is a natural potential of X, $f \in \mathscr{S}$ and $u \leqq f$, and if in addition $u(X_t)_- \leqq f(X_t)$ for all t such that $X_t = X_{t-}$, then $u = P_R f$ for some exact terminal time R on a possibly larger sample space. We give here a similar representation using an operator of the type discussed in the preceding section, one advantage being that one may remain on the original sample space, using only the fields $(\mathscr{F}_t)$, and another being that the last, somewhat unnatural, condition may be dropped.

THEOREM 3.1. *Let $f \in \mathscr{S}$ be finite off a polar set and let u be a natural potential such that $u \leqq f$. Then there exists a natural exact MF M of X such that $u = \bar{P}_M f$.*

Proof. Let $u = u_B$, B a natural additive functional. Since u is finite, B is a.s. finite on $[0, \zeta)$, and by [1], IV, (4.29), if T is a stopping time which is accessible on Λ, then $B_T - B_{T-} = u(X_T)_- - u(X_T)$ a.s. on $\Lambda \cap \{T < \zeta\}$. For every $\varepsilon > 0$, let

$$A_t^\varepsilon = \int_0^t (f(X_s)_- + \varepsilon - u(X_s))^{-1} dB_s .$$

Clearly A^ε is a finite natural AF of X, and if T is an accessible stopping time, $A_T^\varepsilon - A_{T-}^\varepsilon = (f(X_T)_- + \varepsilon - u(X_T))^{-1}(u(X_T)_- - u(X_T))$ a.s. on $\{T < \zeta\}$ and so $A_T^\varepsilon - A_{T-}^\varepsilon < 1$ for any accessible T. There exists therefore a right continuous natural MF, M^ε, such that $S = \zeta$ and

$$(M_{t-}^\varepsilon)^{-1}(-dM_t^\varepsilon) = dA_t^\varepsilon , \qquad t < \zeta .$$

Let $C_t = B_t^c$, the continuous part of B. Then for $t < \zeta$

$$\begin{aligned}
M_t^\varepsilon = \exp&\left\{-\int_0^t [f(X_s)_- + \varepsilon - u(X_s)]^{-1} dC_s\right\} \\
&\times \prod_{s \leqq t} [1 - (f(X_s)_- + \varepsilon - u(X_s))^{-1} \Delta B_s]
\end{aligned}$$

and it is clear that a.s., M_t^ε decreases as ε decreases for all $t \geqq 0$.

Let $M_t = \lim_{(\varepsilon\to 0)} M_t^\varepsilon$, $S = \inf\{t > 0 : M_t = 0\}$. We propose to show that M is a MF of the type considered in the second section. Obviously M is adapted, multiplicative, a.s. decreasing, $M_\zeta = 0$, $M_t M_S \circ \theta_t = M_{t+S\circ\theta_t}$, but it may well happen that $M_S > 0$. Upon taking the monotonic limit as $\varepsilon \to 0$ in the above representation, one sees that

$$(3.2)\qquad \begin{aligned} M_t = \exp\left\{-\int_0^t [f(X_s)_- - u(X_s)]^{-1} dC_s\right\} \\ \times \prod [1 - (f(X_s)_- - u(X_s))^{-1} \varDelta B_s] \end{aligned}$$

for all $t < \zeta$, and from (3.2) one finds

$$(3.3)\qquad S = \inf\left\{t > 0 \colon \int_0^t [f(X_s)_- - u(X_s)]^{-1} dB_s = \infty\right\}.$$

REMARK. In the product term of (3.2), we take

$$[f(X_s)_- - u(X_s)]^{-1} \varDelta B_s = 0 \quad \text{if} \quad \varDelta B_s = 0\,.$$

It is almost surely true that if $M_t > 0$, $M_s^\varepsilon / M_s \leqq M_t^\varepsilon / M_t$ for all $s \leqq t$ whence $M_s^\varepsilon \to M_s$ uniformly on $[0, t]$ if $M_t > 0$. The right continuity of M on $[0, S)$ follows immediately.

To see that M is natural, use (3.2) to observe that on $[0, S)$, the only jumps of M must occur at jump times of B, and that on $\{M_S < M_{S-}, S < \zeta\}$, $\varDelta B_S > 0$, implying that S is accessible on $\{M_{S-} > M_S\}$.

The exactness of M is a consequence of [1], III, (5.9) once it is established that if $P^x\{S = 0\} = 1$, then $E^x \tilde{M}_{v-t} \circ \theta_t \to 0$ as $t \to 0$, for all $v > 0$. However, $\tilde{M} \leqq M$ and it is easy to see that $t \to M_{v-t} \circ \theta_t$ is an increasing function. Because of the monotonic convergence of M_t^ε to M_t, it is legal to interchange limits to obtain

$$\begin{aligned} \lim_{(t\to 0)} M_{v-t} \circ \theta_t &= \lim_{(t\to 0)} \lim_{(\varepsilon\to 0)} M_{v-t}^\varepsilon \circ \theta_t \\ &= \lim_{(\varepsilon\to 0)} \lim_{(t\to 0)} M_{v-t}^\varepsilon \circ \theta_t = \lim_{(\varepsilon\to 0)} M_v^\varepsilon = 0 \text{ a.s. } P^x\,, \end{aligned}$$

using the exactness of M^ε.

We remark at this point that $f(X_S) = u(X_S)$ a.s. on $\{S < \zeta\}$, for by (3.3), on $\{S < \zeta\}$, either $\varDelta B_S > 0$ and $f(X_S)_- = u(X_S)$ or $\varDelta B_S = 0$. In the first case, S is accessible on $\{\varDelta B_S > 0\}$ and so $f(X_S) \leqq f(X_S)_- = u(X_S) \leqq f(X_S)$ whence $u(X_S) = f(X_S)$. In case $\varDelta B_S = 0$, $t \to A_t = \int_{(0,t]} [f(X_s)_- - u(X_s)]^{-1} dB_s$ is left continuous at S. If $A_S = \infty$ and $T_n = \inf\{t > 0 \colon A_t \geqq n\}$ then T_n increases to S a.s. on $\{S < \zeta\}$ and $T_n < S$ a.s. on $\{0 < S < \zeta\}$. Thus S is accessible on $\{A_S = \infty, 0 < S < \zeta\}$ and a.s. on $\{A_S = \infty, 0 < S < \zeta\}$, $\liminf_{(t\to S-)} [f(X_t)_- - u(X_t)] = 0$ which implies $f(X_S)_- = u(X_S)$. But $u(X_S)_- = u(X_S)$ since $\varDelta B_S = 0$ and $f(X_S) \leqq f(X_S)_-$ since S is accessible. This shows that $u(X_S) = f(X_S)$ a.s. on

$\{0 < S < \zeta, A_S = \infty\}$. On $\{S < \zeta, A_S < \infty\}$, one sees from (3.3) that $\liminf_{(t\to S+)}[f(X_t)_- - u(X_t)] = 0$, whence $f(X_S) = u(X_S)$, proving finally that $u(X_S) = f(X_S)$ a.s. on $\{S < \zeta\}$.

From (3.2), we find that a.s. on $[0, S)$

$$(-dM_t)(f(X_t)_- - u(X_t)) = M_{t-}dB_t$$

and a.s. on $\{S < \zeta\}$

$$(M_{S-} - M_S)f(X_S)_- = (M_{S-} - M_S)u(X_S) + M_{S-}\Delta B_S .$$

Thus

$$\begin{aligned}\int_{(0,\zeta)} f(X_t)_-(-dM_t) &= \int_{(0,S)} f(X_t)_-(-dM_t) \\ &\quad + [f(X_S)_-(M_{S-} - M_S) + f(X_S)M_S]1_{\{S<\zeta\}} \\ &= \int_{(0,S)} u(X_t)(-dM_t) + \int_{(0,S)} M_{t-}dB_t \\ &\quad + [(M_{S-} - M_S)u(X_S) + M_{S-}\Delta B_S + u(X_S)M_S]1_{\{S<\zeta\}} \\ &= \int_0^\infty u(X_t)(-d\tilde{M}_t) + \int_0^\infty \tilde{M}_{t-}dB_t .\end{aligned}$$

Since $u(X_T)1_{\{T<\infty\}} = E^x\{(B_\infty - B_T)1_{\{T<\infty\}} \mid \mathscr{F}_T\}$ for all stopping times T, Meyer's integration lemma ([2], VII, T. 15) applies to give

$$E^x\int_0^\infty u(X_t)(-d\tilde{M}_t) = E^x\int_0^\infty (B_\infty - B_t)(-d\tilde{M}_t) .$$

Thus, for $x \in E_M$,

$$\begin{aligned}\bar{P}_M f(x) &= E^x\int_0^\infty (B_\infty - B_t)(-d\tilde{M}_t) + \int_0^\infty \tilde{M}_{t-}dB_t \\ &= E^x(B_\infty) + E^x\int_0^\infty \tilde{M}_t dB_t - \int_0^\infty B_t(-d\tilde{M}_t) \\ &= u(x)\end{aligned}$$

upon integrating by parts.

If $x \notin E_M$, $\bar{P}_M f(x) = f(x) = E^x f(X_S) = E^x u(X_S) = u(x)$, and the theorem is completely proven.

4. REMARKS. It is natural to ask for a specification of the class $\{\bar{P}_M^\alpha f\colon M$ a natural exact $MF\}$, for a given $f \in \mathscr{S}$. The following example shows that although it contains f and all natural potentials, it need not include all excessive functions dominated by f. Let X be uniform motion to the right on the real line, let $f \equiv 1$ and $u \equiv 1/2$. Obviously $\bar{P}_M f(x) = P_{\tilde{M}}1(x)$ for all x, and because we can write down (up to equivalence) the form of $\tilde{M}$, it is a simple matter to check that $P_{\tilde{M}}1 = 1/2$ has no solution for $\tilde{M}$.

A particular example of an operator $\bar{P}_M$ which may be of interest is obtained by taking, for a fixed Borel subset B of E,

$$M_t = 1_{[0,T_B \wedge \zeta)}(t) + 1_{\{t=T_B<\zeta, X_{t-}\neq X_t\}} .$$

Then $S = \inf\{t > 0: M_t = 0\} = T_B \wedge \zeta$, and using the fact that S is totally inaccessible on $\{X_S \neq X_{S-}, S < \zeta\}$, $P^x\{t = T_B < \zeta, X_{t-} \neq X_t\} = 0$ for all $t \geqq 0$ and $x \in E$. It follows readily that M is a MF satisfying (2.1). Define, for $f \in \mathscr{S}$,

$$\bar{P}_B f(x) = \bar{P}_M f(x) = E^x\{f(X_{T_B})_-; T_B < \zeta, X_{T_B} = X_{T_B-}\} + E^x\{f(X_{T_B}); T_B < \zeta, X_{T_B} \neq X_{T_B-}\} .$$

Because of Theorem (2.7), $\bar{P}_B f \in \mathscr{S}$ if $f \in \mathscr{S}$.

One simple use of the operator $\bar{P}_B$ is afforded by the following example. Let B be a finely closed Borel subset of E and let f be a uniformly integrable excessive function. Assume that X is a Hunt process. Let f^B be the lower envelope of the family of excessive functions which dominate f on a (variable) neighborhood of B. In [1], VI, (2.12)-(2.15), it is shown, under different hypotheses, that $f^B = P_B f$ off a certain exceptional set provided f is "admissible". However, under the hypotheses given above without assuming f to be admissible, it is a simple matter, using [1], I, (11.3) together with certain facts from [1], VI, (2.12)-(2.15), to obtain $\bar{P}_B f \leqq f^B$ everywhere, and $\bar{P}_B f(x) = f^B(x)$ except possibly on $B - B^r$. It does not seem to be easy to remove the restrictions imposed above to obtain a general representation of f^B.

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