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TWO PRIMARY FACTOR INEQUALITIES

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In the theory of integral functions, the expressions

$$(1) \qquad E(z, p) = (1 - z) \exp \left\{ \sum_1^p \frac{z^r}{r} \right\}, \; p = 1, 2, \cdots$$

called *primary factors*, are of some importance, and it is of interest to find upper bounds for $|E(z, p)|$. Clearly $E(z, p) = 0$ only for $z = 1$, and so for other values, define $f(z, p) = \log |E(z, p)|$. It is known that for suitable constants a_p, b_p the inequalities

$$(2) \qquad f(z, p) \leqq a_p |z|^p, \; |z| \geqq 1, z \neq 1$$

$$(3) \qquad f(z, p) \leqq b_p |z|^{p+1}, \; |z| \leqq 1, z \neq 1$$

are satisfied; for instance Hille has shown that one may take $a_p = 1 + \sum_1^p 1/r \leqq 2 + \log p$ and $b_p = 1$.

In this paper, the smallest values of both a_p and b_p are determined, the latter in closed form.

Throughout, we shall write $z = \rho e^{i\theta}$, where without loss of generality $\rho \geqq 0$, $0 \leqq \theta \leqq \pi$. Then

$$(4) \qquad f(z, p) = \frac{1}{2} \log (1 - 2\rho \cos \theta + \rho^2) + \sum_1^p \frac{\rho^r}{r} \cos r\theta \, .$$

Also, using the Taylor series for $\log (1 - z)$ gives from (1)

$$(5) \qquad f(z, p) = -\rho^{p+1} \sum_0^\infty \frac{\rho^r}{p + r + 1} \cos (p + r + 1)\theta \, ,$$

provided $\rho < 1$. A further expression is obtained by writing $\log E(z, p)$ as an integral of its derivative and taking real parts, to give

$$f(z, p) = \int_0^\rho \frac{t \cos p\theta - \cos (p + 1)\theta}{1 - 2t \cos \theta + t^2} t^p \, dt \, ,$$

provided $\theta \neq 0$ or $\rho < 1$.

The problem considered in this paper is the determination of the maxima of the functions

$$(6) \qquad g(z, p) = \rho^{-p} f(z, p) \text{ for } \rho \geqq 1$$

and

$$h(z, p) = \rho^{-p-1} f(z, p) \text{ for } \rho \leqq 1 \, ,$$

and to show where these occur.

1. Summary of results. Henceforth we use a_p and b_p to denote the smallest constants for which (2) and (3) hold. We shall show that both a_p and b_p are monotone decreasing functions of p. The value of a_1 is given by $a_1 = \log(\rho - 1)$ where ρ is the solution of the transcendental equation $(\rho - 1)\log(\rho - 1) = \rho, \rho > 1$ and the maximum occurs at $z = \rho$. Also $a_2 = 1$, the maximum occuring at $z = 2$, and a_∞ is given by the common value of x^{-1} and

$$e^{-x}\left(\gamma + \int_0^1 \frac{e^s - 1}{s} ds + \int_1^x \frac{e^s}{s} ds\right),$$

for the unique value of x which makes these expressions equal, γ denoting Euler's constant. For each $p \geqq 2$, the maximum occurs at a point z on the real axis which satisfies $1 < z \leqq 2$.

The z maximizing b_p occur on $|z| = 1$, with $\theta = \pi/(2p + 1)$, $p = 1, 2, 3, \cdots$. For $p > 1$ the maximum is unique, but for $p = 1$ it is attained at every point of the arc $|z - 1| = 1, |z| \leqq 1$. We derive the explicit bounds

$$\frac{1}{2} \geqq b_p > \log \pi/2 + \gamma - \int_0^{\pi/2} \frac{1 - \cos x}{x} dx,$$

and both bounds are sharp. We also have an explicit formula

$$b_p = \log\left(2 \sin \frac{1}{2}\theta\right) + \sum_1^p \frac{1}{r} \cos r\theta,$$

where $\theta = \pi/(2p + 1)$. In particular these results give

$$1.2785 \geqq a_p > 0.7423, \frac{1}{2} \geqq b_p > 0.4719.$$

Since $a_2 = 1$ we have therefore

$$\log|E(z, p)| \leqq \min(|z|^p, |z|^{p+1}), p = 2, 3, \cdots,$$

and this is sharp.

The numerical values of a_p and b_p are as follows.

p	a_p	b_p
1	1.2785	0.5000
2	1.0000	0.4823
3	0.9123	0.4771
4	0.8691	0.4752
5	0.8435	0.4741
∞	0.7423	0.4719

2. Preliminaries. It is clear that for (2) and (3) to hold, both a_p and b_p must be positive, since for example $z = 2$ and $z = \varepsilon \exp i\pi/(p+1)$ with ε sufficiently small give positive values of $f(z, p)$. We see therefore that not only the point $z = 1$, but also a neighbourhood of this point can be excluded from the discussion. We find by elementary means that

$$(8)\quad \frac{\partial g}{\partial \theta} = \frac{\rho\{\sin(p+1)\theta - \rho \sin p\theta\}}{1 - 2\rho\cos\theta + \rho^2}$$

$$(9)\quad \frac{\partial^2 g}{\partial \theta^2} = \frac{\partial\{(p+1)\cos(p+1)\theta - p\rho\cos p\theta\}}{1 - 2\rho\cos\theta + \rho^2} - \frac{2\rho\sin\theta}{1 - 2\rho\cos\theta + \rho^2}\frac{\partial g}{\partial \theta}$$

$$(10)\quad \frac{\partial g}{\partial \rho} = -\frac{pg}{\rho} + \frac{\rho\cos p\theta - \cos(p+1)\theta}{1 - 2\rho\cos\theta + \rho^2}$$

$$(11)\quad \frac{\partial h}{\partial \theta} = \frac{\sin(p+1)\theta - \rho\sin p\theta}{1 - 2\rho\cos\theta + \rho^2}$$

$$(12)\quad \frac{\partial^2 h}{\partial \theta^2} = \frac{(p+1)\cos(p+1)\theta - p\rho\cos\theta}{1 - 2\rho\cos\theta + \rho^2} - \frac{2\rho\sin\theta}{1 - 2\rho\cos\theta + \rho^2}\frac{\partial h}{\partial \theta}$$

$$(13)\quad \frac{\partial h}{\partial \rho} = -\frac{(p+1)h}{\rho} + \frac{\rho\cos p\theta - \cos(p+1)\theta}{\rho(1 - 2\rho\cos\theta + \rho^2)}\,.$$

3. The case $\rho = 1$. We consider first the unit circle on which of course f, g and h coincide, with $0 \leqq \theta \leqq \pi$. Then by (8) we find that $\partial f/\partial\theta = 1/2\cos(p + (1/2))\theta \operatorname{cosec}(1/2)\theta$, and so local maxima occur at $\theta = \beta, 5\beta, 9\beta, \cdots$ where $\beta = \pi/(2p+1)$. We shall show that $f(\beta) > f(5\beta) > f(9\beta) > \cdots$ and hence that $f(\beta)$ is the largest value taken by $f(z, p)$ on $|z| = 1$. For, let $n \geqq 0$ with $(4n+5)\beta \leqq \pi$. Then

$$f((4n+5)\beta) - f((4n+1)\beta) = \int_{(4n+1)\beta}^{(4n+5)\beta} f'(\theta)d\theta$$

$$= \frac{1}{2}\int_{(4n+1)\beta}^{(4n+3)\beta} \cos\frac{\pi\theta}{2\beta}\operatorname{cosec}\frac{1}{2}\theta d\theta + \frac{1}{2}\int_{(4n+3)\beta}^{(4n+5)\beta} \cos\frac{\pi\theta}{2\beta}\operatorname{cosec}\frac{1}{2}\theta d\theta$$

$$= \frac{-\beta}{2\pi}\int_0^{2\pi} \frac{\sin\frac{1}{2}\phi}{\sin\left(4n + 3 - \frac{\phi}{\pi}\right)\frac{\beta}{2}} d\phi + \frac{\beta}{2\pi}\int_0^{2\pi} \frac{\sin\frac{1}{2}\phi}{\sin\left(4n + 3 + \frac{\phi}{\pi}\right)\frac{\beta}{2}} d\phi$$

$$< 0\,,$$

where we have substituted $\phi = (4n+3)\pi - \pi\theta/\beta$ in the first integral, and $\phi = -(4n+3)\pi + \pi\theta/\beta$ in the second.

Thus we obtain in view of (4), that for $|z| = 1$,

$$(14)\qquad f(z, p) \leqq \sigma_p = \log\left(2\sin\frac{\pi}{4p+2}\right) + \sum_1^p \frac{1}{r}\cos\frac{r\pi}{2p+1}\,.$$

We now consider σ_p, and prove first that $\sigma_p > \sigma_{p+1}$. Define δ by $\pi = 2(2p+1)(2p+3)\delta$. Then

$$\begin{aligned}\sigma_p - \sigma_{p+1} &= \log \frac{\sin(2p+3)\delta}{\sin(2p+1)\delta} + \sum_1^p \frac{1}{r}\{\cos 2(2p+3)r\delta - \cos 2(2p+1)r\delta\} \\ &\quad - \frac{1}{p+1}\cos 2(2p+1)(p+1)\delta \\ &= \lambda(\delta), \text{ say.}\end{aligned}$$

Thus if $\lambda(\phi)$ is defined for $0 < \phi \leqq \delta$ by the same formula with δ replaced by ϕ, we find that as $\phi \to 0$,

$$\lambda(\phi) \to \log \frac{2p+3}{2p+1} - \frac{1}{p+1} > 0 .$$

Also

$$\begin{aligned}\lambda'(\phi) &= \{(2p+3)\operatorname{cosec}(2p+3)\phi - (2p+1)\operatorname{cosec}(2p+1)\phi\}\cos\frac{\pi\phi}{2\delta} \\ &> 0 ,\end{aligned}$$

since $x \operatorname{cosec} x$ is strictly increasing in $(0, \pi/2)$.

Thus $\lambda(\delta) > 0$, and so

$$\sigma_p > \sigma_{p+1} . \tag{15}$$

Also as $p \to \infty$ we find that

$$\begin{aligned}\sigma_p &= \log\left(2\sin\frac{\pi}{4p+2}\right) + \sum_1^p \frac{1}{r}\cos\frac{r\pi}{2p+1} \\ &= \log\frac{\pi}{2p+1} + o(1) + \sum_1^p \frac{1}{r} + \sum_1^p \frac{1}{r}\left\{\cos\frac{r\pi}{2p+1} - 1\right\} \\ &= \log \pi/2 + \left\{\sum_1^p \frac{1}{r} - \log p\right\} - \int_0^{\pi/2} \frac{1-\cos x}{x}dx + o(1) \\ &\to \log\frac{1}{2}\pi + \gamma - \int_0^{\pi/2}\frac{1-\cos x}{x}dx = 0.4719 .\end{aligned}$$

Thus we find, since $\sigma_1 = 1/2$, that for all p

$$\frac{1}{2} \geqq \sigma_p > 0.4719 . \tag{16}$$

4. The case $\rho \leqq 1$. For $\rho \leqq 1$, we consider first $p = 1$, where the situation is slightly different from the remaining values of p. Using (11) we see that if $\rho \neq 1$, then for fixed ρ, h has turning values, regarded as a function of θ, only for $\theta = 0, \theta = \pi$ and $2\cos\theta = \rho$. Using (12) we find that both $\theta = 0$ and $\theta = \pi$ give minima, and so for each $\rho \in (0, 1)$ we find that

$$h(z, 1) \leqq \rho^{-2}\Big(\frac{1}{2}\log 1 + \frac{1}{2}\rho^2\Big) = \frac{1}{2}\,,$$

with equality if and only if $2\cos\theta = \rho$. Thus we have $b_1 = 1/2$ with equality attained at every point of the arc $|z - 1| = 1, |z| \leqq 1$.

For $p \geqq 2$, the situation is quite different. Clearly whatever b_p with turn out to be, there will be equality in (3) for $z = 0$. But for $0 < \rho < 2/7$, we find using (5) and (7) that

$$\begin{aligned} h(z, p) &= -\sum_0^\infty \frac{\rho^r}{p + r + 1}\cos(p + r + 1)\theta \\ &< \frac{1}{3}\sum_0^\infty\Big(\frac{2}{7}\Big)^r = \frac{7}{15} < \sigma_p, \text{ by (16)}\,. \end{aligned}$$

Thus the maximum of $h(z, p)$ occurs in the closed annulus $2/7 \leqq \rho \leqq 1$.

Again consider a fixed value of $p < 1$. By (11) the greatest value of h, regarded as a function of θ, occurs at a solution of $\sin(p + 1)\theta = \rho\sin p\theta$. $\theta = 0$ is impossible since then $\partial^2 h/\partial\theta^2 > 0$ by (12) and $\theta = \pi$ can be neglected since then by (5) and (7) we get

$$h(-\rho, p) = \sum_0^\infty \frac{\rho^r}{p + r + 1}(-1)^{p+r} \leqq \frac{1}{p + 1} \leqq \frac{1}{3} < \sigma_p\,.$$

A glance at the sketch of $y = \sin(p + 1)x/\sin px$ for $x \in (0, \pi)$, shown in Figure 1, reveals that there are precisely p other values of θ to consider, since it is readily shown that each branch of the curve is monotone strictly decreasing. Again we consider the sign of $\partial^2 h/\partial\theta^2$. Since $\rho < 1$ we find that for given ρ, the intersection of $y = \rho$ with the rth. branch of the curve satisfies

$$\frac{2r - 1}{2p + 1}\pi < \theta < \frac{r\pi}{p + 1}\,,$$

whence $p\theta \in ((r - 1)\pi, r\pi)$ and $(p + 1)\theta \in ((r - (1/2))\pi, r\pi)$. Thus at such a point we find from (12) and substituting for ρ,

$$\frac{\partial^2 h}{\partial^2\theta} = \frac{\sin p\theta}{\sin^2\theta}\{\sin p\theta\cos(p + 1)\theta - p\sin\theta\}$$

and so the second factor is negative. Thus $\partial^2 h/\partial\theta^2 < 0$ only if $\sin p\theta > 0$, i.e. if r is odd. Moreover at a local maximum we have using (13)

$$h = \frac{\sin p\theta}{(p + 1)\sin\theta}\,.$$

Thus if $\theta \leqq \pi/2$ we we find that except on the first branch $\theta > 2\pi/p$ and so

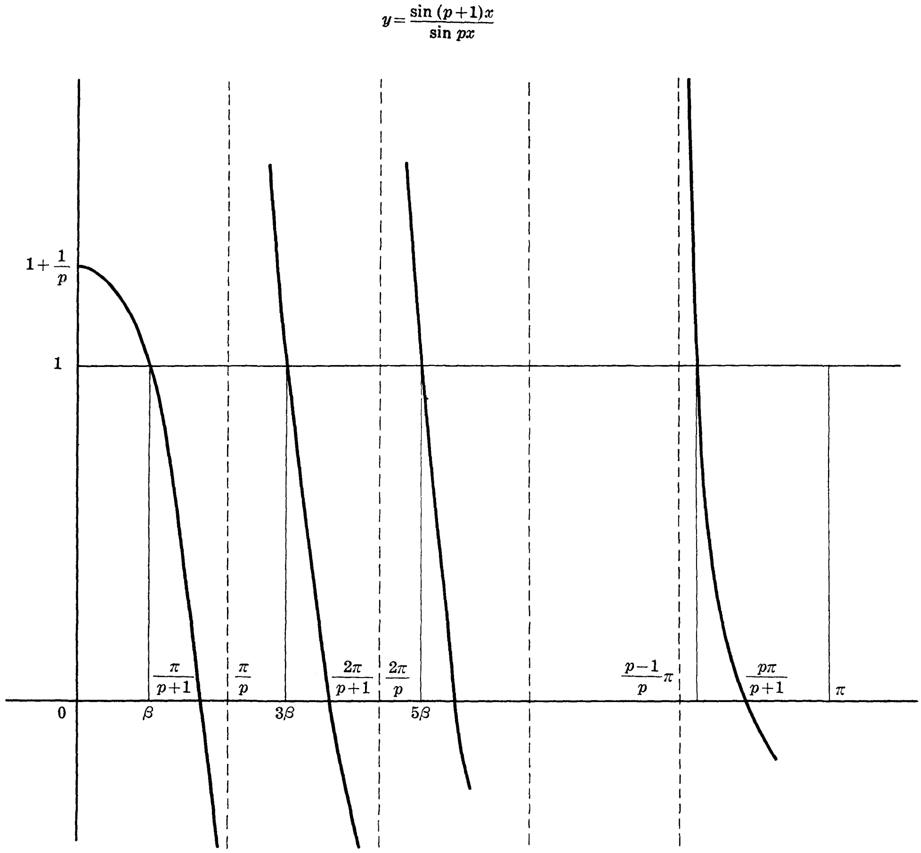

FIGURE 1

$$\begin{aligned} h &\leqq \frac{1}{(p+1)\sin\theta} \\ &\leqq \frac{\pi}{2(p+1)\theta} \\ &\leqq \frac{p}{4(p+1)} < \sigma_p \,. \end{aligned}$$

Similarly if $\theta \geqq \pi/2$ we find that

$$\pi - \theta \geqq \pi - \frac{\pi p}{p+1} = \frac{\pi}{p+1}$$

and so

$$\begin{aligned} h &\leqq \frac{1}{(p+1)\sin\theta} \\ &\leqq \frac{1}{p+1}\operatorname{cosec}\frac{\pi}{p+1} \\ &\leqq \frac{1}{3}\operatorname{cosec}\frac{1}{3}\pi < \sigma_p \,, \end{aligned}$$

since x cosec x increases over $(0, \pi/2)$.

Thus we need only consider the first branch. Let $k(\rho)$ be the value taken by $h(z, p)$ with $\rho \in (0, 1)$ and θ defined by $\rho \sin p\theta = \sin (p+1)\theta$, $\theta \in (0, \pi/(p+1))$. Then

$$\begin{aligned} \frac{dk}{d\rho} &= \frac{\partial h}{\partial \rho} + \frac{\partial h}{\partial \theta}\Big/\frac{d\rho}{d\theta} = \frac{\partial h}{\partial \rho} \\ &= -\frac{(p+1)k}{\rho} + \frac{1}{\rho}\frac{\sin p\theta}{\sin \theta} \,. \end{aligned} \tag{17}$$

Thus

$$\begin{aligned} \frac{d}{d\rho}\left\{\rho\frac{dk}{d\rho} + (p+1)k\right\} &= \frac{d}{d\theta}\left(\frac{\sin p\theta}{\sin \theta}\right)\Big/\frac{d}{d\theta}\left(\frac{\sin (p+1)\theta}{\sin p\theta}\right) \\ &> 0 \,, \end{aligned}$$

since both $(\sin p\theta)/(\sin \theta)$ and $(\sin (p+1)\theta)/(\sin p\theta)$ decrease over $(0, \pi/(p+1))$. Therefore

$$\frac{d}{d\rho}\left\{\rho^{p+2}\frac{dk}{d\rho}\right\} = \rho^{p+1}\frac{d}{d\rho}\left\{\rho\frac{dk}{dp} + (p+1)k\right\} > 0 \,,$$

and so $\rho^{p+2}\, dk/d\rho$ increases. But as $\rho \to 0$, $\theta \to \pi/(p+1)$, and so using (17) we see that $\rho^{p+2}\, dk/d\rho \to 0$. Thus for $\rho > 0$, $dk/d\rho > 0$, whence k increases over $(0, 1)$. Thus for all such z, $h(z, p) \leqq k(1) = \sigma_p$ with equality if and only if $z = \exp i\pi/(2p+1)$. This concludes the discussion of this case.

5. The case $\rho \geqq 1$. We find that

$$\frac{\partial}{\partial \theta}\left\{\frac{1}{2}\log(1 - 2\rho\cos\theta + \rho^2) + \rho\cos\theta\right\} = \frac{\rho^2 \sin\theta(2\cos\theta - \rho)}{1 - 2\rho\cos\theta + \rho^2}$$

and so if $\rho \geqq 2$, $1/2 \log(1 - 2\rho\cos\theta + \rho^2) + \rho\cos\theta \leqq \log(\rho - 1) + \rho$, whence for $\rho \geqq 2$,

$$\begin{aligned} g(z, p) &= \rho^{-p}\left\{\frac{1}{2}\log(1 - 2\rho\cos\theta + \rho^2) + \sum_1^p \frac{\rho^r}{r}\cos r\theta\right\} \\ &\leqq \rho^{-p}\left\{\log(\rho - 1) + \sum_1^p \frac{\rho^r}{r}\right\} = g(\rho, p) \,. \end{aligned}$$

Also $g(\rho, 2)$ is decreasing for $\rho > 2$, for by (10) we find that

$$\frac{dg(\rho, 2)}{d\rho} = \frac{-2}{\rho^3}\log(\rho - 1) - \frac{\rho - 2}{\rho^2(\rho - 1)} \,.$$

But we now see from the definition of $g(z, p)$ that

$$g(\rho, p+1) = \frac{1}{p+1} + \frac{1}{\rho}g(\rho, p)$$

and so by induction we see that $g(\rho, p)$ decreases for $\rho \geqq 2$ for each $p \geqq 2$. Thus

$$\text{(18)} \qquad \text{for } \rho \geqq 2, p \geqq 2 \;\; g(z, p) \leqq g(2, p)\,,$$

with equality only for $z = 2$.

Consider first the case $p = 1$. If $\rho \leqq 2$, we find that for given ρ, g is greatest when $2\cos\theta = \rho$, or $g(z, 1) \leqq \rho/2 \leqq 1$ for $\rho \leqq 2$. For $2 \leqq \rho$, we know that $g(z, 1) \leqq g(\rho, 1) = \rho^{-1}\log(\rho - 1) + 1$, and it is easily seen that this expression has precisely one turning value, and that a maximum, which occurs where $(\rho - 1)\log(\rho - 1) = \rho$: This gives $\rho = 4.5911$ and then $g(\rho, 1) = 1.2785$. Thus $a_1 = 1.2785$.

Secondly, consider $p = 2$. For $\rho \leqq 2$ we have

$$\begin{aligned}\rho^2 g(z, 2) &= \frac{1}{2}\log(1 - 2\rho\cos\theta + \rho^2) + \rho\cos\theta + \frac{1}{2}\rho^2\cos 2\theta \\ &\leqq \frac{1}{2}\rho^2 + \frac{1}{2}\rho^2\cos 2\theta, \text{ as before} \\ &\leqq \rho^2\,,\end{aligned}$$

where equality occurs only if $\rho = 2\cos\theta$ and $\cos 2\theta = 1$ are satisfied simultaneously; this does occur and at the single point $z = 2$. Thus $a_2 = 1$.

Finally we consider $p \geqq 3$, and then in view of (18) we need only consider the annulus $1 \leqq |z| \leqq 2$. At a local maximum, we obtain from (8), $\rho\sin p\theta = \sin(p + 1)\theta$. In view of (9) $\theta = 0$ arises only if $\rho \geqq 1 + p^{-1}$, since otherwise $\partial^2 g/\partial\theta^2$ is positive. $\theta = \pi$ can be dismissed, since by (10) a local maximum at such a point would give $g \leqq p^{-1} \leqq 1/3 < \sigma_p$, by (16). Referring to the figure, we find therefore that we need to consider three cases

(a) $\theta = 0$ for $\rho \geqq 1 + p^{-1}$,

(b) $0 < \theta \leqq \pi/(2p + 1)$ for $1 \leqq \rho \leqq 1 + p^{-1}$,

(c) values of θ between π/p and $\pi - \pi/(p + 1)$.

As before the final case can be dismissed, since at such a local maximum we find from (10) that

$$\begin{aligned} g &= \frac{\sin(p + 1)\theta}{p\sin\theta} \\ &\leqq p^{-1}\operatorname{cosec}\frac{\pi}{p + 1} \\ &= \frac{p + 1}{\pi p}\,\frac{\pi}{p + 1}\operatorname{cosec}\frac{\pi}{p + 1} \\ &\leqq \frac{p + 1}{p}\,\frac{1}{4}\operatorname{cosec}\frac{1}{4}\pi \\ &\leqq \frac{1}{3}\operatorname{cosec}\frac{1}{4}\pi = \frac{1}{3}2^{1/2} < \sigma_p, \text{ in view of (16)}\,. \end{aligned}$$

Now in the second case, let $m(\rho)$ be the value taken by $g(z, p)$ when $\rho \sin p\theta = \sin (p+1)\theta$ and $0 < \theta \leqq \pi/(2p+1)$. Then using (8) and (10) we obtain similarly to (17),

$$\frac{dm}{d\rho} = \frac{-pm}{\rho} + \frac{\sin p\theta}{\sin \theta}, \tag{19}$$

and so

$$\begin{aligned} \frac{d}{d\rho}\left\{\rho^{p+1}\frac{dm}{d\rho}\right\} &= \rho^p \frac{d}{d\rho}\left\{\rho\frac{dm}{d\rho} + pm\right\} \\ &= \rho^p \frac{d}{d\theta}\left\{\frac{\sin (p+1)\theta}{\sin \theta}\right\} \Big/ \frac{d}{d\theta}\left\{\frac{\sin (p+1)\theta}{\sin p\theta}\right\} \\ &> 0, \text{ as before.} \end{aligned}$$

Thus $\rho^{p+1}\, dm/d\rho$ increases as ρ increases from 1 to $1 + p^{-1}$. But using (19) we see that when $\rho = 1$,

$$\begin{aligned} \frac{dm}{d\rho} &= -p\sigma_p + \sin \frac{p\pi}{2p+1} \Big/ \sin \frac{\pi}{2p+1} \\ &= -p\sigma_p + \frac{1}{2}\operatorname{cosec} \frac{\pi}{4p+2} \\ &> -p\sigma_p + (2p+1)/\pi \\ &> p(2/\pi - \sigma_p) > 0, \text{ in view of (16).} \end{aligned}$$

Thus $m(\rho)$ is an increasing function of ρ as ρ increases from 1 to $1 + p^{-1}$, and in particular $g(1 + p^{-1}, p) \geqq g(z, p)$ for $|z| \leqq 1 + p^{-1}$. Thus we need only consider case (a).

Let

$$\begin{aligned} \Delta_p &= f(1 + p^{-1}, p) \\ &= -\log p + \sum_1^p \frac{1}{r}\left(1 + \frac{1}{p}\right)^r \\ &= -\log p + \sum_1^p \int_0^{1+p^{-1}} t^{r-1}\, dt \\ &= -\log p + \int_0^{1+p^{-1}} \frac{t^p - 1}{t - 1} dt\,. \end{aligned} \tag{20}$$

Thus

$$\begin{aligned}
\Delta_{p+1} - \Delta_p &= -\log\frac{p+1}{p} + \int_0^{1+(p+1)^{-1}} \frac{t^{p+1}-t^p}{t-1}dt \\
&\quad - \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{t^p - 1}{t-1}dt \\
&= -\log\frac{p+1}{p} + \frac{1}{p+1}\left\{1+\frac{1}{p+1}\right\}^{p+1} + \log\frac{p+1}{p} \\
&\quad - \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{t^p}{t-1}dt \\
&= \frac{1}{p+1}\left\{1+\frac{1}{p+1}\right\}^{p+1} - I, \text{ say .}
\end{aligned}$$

To estimate I we observe that for $t > 1 + (p+1)^{-1}$,

$$\frac{d}{dt}\left(\frac{t^{p+2}}{t-1}\right) = \frac{(p+1)t^{p+1}}{(t-1)^2}\left(t - 1 - \frac{1}{p+1}\right) > 0 ,$$

and so

$$\begin{aligned}
I &= \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{t^{p+2}}{t-1}\frac{dt}{t^2} \\
&> (p+1)\left\{1+\frac{1}{p+1}\right\}^{p+2}\int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{dt}{t^2} \\
&= (p+2)\left\{1+\frac{1}{p+1}\right\}^{p+1}\left\{\frac{p+1}{p+2} - \frac{p}{p+1}\right\} \\
&= \frac{1}{p+1}\left\{1+\frac{1}{p+1}\right\}^{p+1} ,
\end{aligned}$$

and so

$$\Delta_{p+1} < \Delta_p . \tag{21}$$

From (10) we see that if $\theta = 0$, $\partial g/\partial\rho = -pg/\rho + (\rho - 1)^{-1}$, and so it is easily verified that $\partial g/\partial\rho > 0$ at $z = 1 + p^{-1}$, and that $\partial g/\partial\rho < 0$ at $z = 2$. Thus there exists at least one turning value of g on the real axis between these two points. At such a point $g = \rho/p(\rho - 1)$ and so

$$\begin{aligned}
\frac{\partial^2 g}{\partial\rho^2} &= -\frac{p}{\rho}\frac{\partial g}{\partial\rho} + \frac{pg}{\rho^2} - \frac{1}{(\rho-1)^2} \\
&= \frac{1}{\rho(\rho-1)} - \frac{1}{(\rho-1)^2} < 0 ,
\end{aligned}$$

and so there is exactly one such turning value, and that a maximum. Now let

$$\mu(x, p) = g(1 + xp^{-1}, p),\ x \geqq 1 . \tag{22}$$

Then using (20) we find

$$\begin{aligned}
\mu(x, p) &= \left(1 + \frac{x}{p}\right)^{-p}\left\{\log\frac{x}{p} + \sum_1^p \frac{1}{r}\left(1 + \frac{x}{p}\right)^r\right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p}\left\{\Delta_p + \log x + \sum_1^p \frac{1}{r}\left\{\left(1 + \frac{x}{p}\right)^r - \left(1 + \frac{1}{p}\right)^r\right\}\right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p}\left\{\Delta_p + \log x + \sum_1^p \int_{1+p^{-1}}^{1+xp^{-1}} t^{r-1}dt\right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p}\left\{\Delta_p + \log x + \int_{1+p^{-1}}^{1+xp^{-1}} \frac{t^p - 1}{t - 1}dt\right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p}\left\{\Delta_p + \int_1^x\left(1 + \frac{s}{p}\right)^p \frac{ds}{s}\right\} \\
&= A_p(x) + B_p(x), \text{ say.}
\end{aligned}$$

Now, $(1 + x/n)^n$ is an increasing sequence and by (21) Δ_n is decresing. Thus $A_{p+1}(x) < A_p(x)$. We shall show that $B_{p+1}(x) \leqq B_p(x)$ too. We find that for $s < x$,

$$\begin{aligned}
\frac{\left(1 + \frac{s}{p}\right)^p}{\left(1 + \frac{x}{p}\right)^p} \div \frac{\left(1 + \frac{s}{p+1}\right)^{p+1}}{\left(1 + \frac{x}{p+1}\right)^{p+1}} &= \frac{(p + s)^p(p + 1 + x)^{p+1}}{(p + x)^p(p + 1 + s)^{p+1}} \\
&= \frac{p + 1 + x}{p + 1 + s}\left\{1 + \frac{x - s}{p^2 + p(x + s + 1) + s + sx}\right\}^{-p} \\
&> \frac{p + 1 + x}{p + 1 + s}\left\{1 - \frac{p(x - s)}{p^2 + p(x + s + 1) + s + sx}\right\} \\
&= \frac{p^2 + p(2s + 1) + s + sx}{p^2 + p(2s + 1) + s + s^2} > 1\,,
\end{aligned}$$

since $(1 + \varepsilon)^{-p} > 1 - p\varepsilon$ for every positive ε. Thus

$$\left(1 + \frac{x}{p}\right)^{-p}\left(1 + \frac{s}{p}\right)^p > \left(1 + \frac{x}{p+1}\right)^{-p-1}\left(1 + \frac{s}{p+1}\right)^{p+1}$$

for $1 \leqq s < x$, and so $B_p(x) \geqq B_{p+1}(x)$.

We see therefore that

$$g\left(1 + \frac{x}{p+1}, p + 1\right) < g\left(1 + \frac{x}{p}, p\right) \leqq a_p\,,$$

and so $a_{p+1} < a_p$.

Also, since $\mu(x, p) > \mu(x, p + 1)$, we see that

$$\mu(x, p) > \mu(x) = \lim_{p\to\infty} \mu(x, p) = e^{-x}\left\{\Delta + \int_1^x s^{-1}e^s ds\right\}\,,$$

where $\Delta = \lim_{p\to\infty} \Delta_p$. Now from (20) we find that

$$\begin{aligned}\Delta_p &= -\log p + \sum_1^p \frac{1}{r} + \sum_1^p \frac{1}{r}\left\{\left(1+\frac{1}{p}\right)^r - 1\right\} \\ &= -\log p + \sum_1^p \frac{1}{r} + \sum_1^p \int_0^1 \frac{1}{p}\left(1+\frac{s}{p}\right)^{r-1} ds\,, \\ &= -\log p + \sum_1^p \frac{1}{r} + \int_0^1 \left\{\left(1+\frac{s}{p}\right)^p - 1\right\}\frac{ds}{s}\,,\end{aligned}$$

and so

$$\Delta = \gamma + \int_0^1 (e^s - 1)s^{-1}ds = 1.895118\,.$$

Now

$$\begin{aligned}\mu'(x) &= -\mu(x) + x^{-1} \\ \mu''(x) &= -\mu'(x) - x^{-2}\,,\end{aligned}$$

and so $\mu(x)$ has precisely one maximum, and at this point $\mu(x) = x^{-1}$, with

$$x^{-1}e^x = \Delta + \int_1^x s^{-1}e^s ds = \Delta + x^{-1}e^x - e + \int_1^x s^{-2}e^s ds\,,$$

or

$$\int_1^x s^{-2}e^s ds = e - \Delta = 0.823164\,,$$

whence $x = 1.3472$ and so $\mu_{\max} = 0.7423$.

Thus we find that since $\mu(x, p) > \mu(x)$, we can always choose x such that $\mu(x, p) > 0.7423$, and so $a_p > 0.7423$. Thus as p increases from 2 to ∞, a_p decreases from 1 to 0.7423.

References

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