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# MULTIPLICITY AND THE AREA OF AN $(n-1)$ CONTINUOUS MAPPING

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**For a class of mappings considered by Goffman and Ziemer [Annals of Math. 92 (1970)] it is shown that the area is given by the integral of a multiplicity function and a convergence theorem is proved.**

**1. Introduction.** A theory of surface area for mappings beyond the class of continuous mappings was initiated in [2]. This theory includes certain essentially discontinuous mappings for which it seems natural that the area be given by the classical integral formula.

Let $Q = R^n \cap \{x: 0 < x_i < 1 \text{ for } 1 \leqq i \leqq n\}$. For each $i \in \{1, \cdots, n\}$ and $r \in I = \{t: 0 < t < 1\}$ let $P_i(r) = Q \cap \{x: x_i = r\}$. A mapping $f: Q \rightarrow R^m$, $n \leqq m$, is said to be $n-1$ continuous if, for each $i, f \mid P_i(r)$ is continuous for almost every (in the sense of 1-dimensional Lebesgue measure) $r \in I$. A sequence $\{f_j\}$ of mappings from $Q$ into $R^m$ is said to converge $n-1$ to $f$ if, for each $i, f_j \mid P_i(r)$ converges uniformly to $f \mid P_i(r)$ for almost every $r \in I$.

The area of an $n-1$ continuous mapping $f: Q \rightarrow R^m$ is defined as

$$A(f) = \inf \varliminf_{j \rightarrow \infty} a(f_j)$$

where the infimum is taken over all sequences $\{f_j\}$ of quasilinear mappings converging $n-1$ to $f$ and $a(f_j)$ denotes the elementary area of $f_j$. In [2] it was shown that $A(f)$ coincides with Lebesgue area if $f$ is continuous.

For real $p \geqq 1$, let $W_p^1(Q)$ denote those functions in $L^p(Q)$ whose distribution first partial derivatives are functions in $L^p(Q)$. Suppose $f: Q \rightarrow R^m$ with $f = (f^1, \cdots, f^m)$ and $f^i \in W_{p_i}^1(Q)$, $p_i > n-1$ for $1 \leqq i \leqq m$ and $\sum_{j=1}^n 1/p_{i_j} \leqq 1$ whenever $1 \leqq i_1 < \cdots < i_n \leqq m$. It was shown in [3] that $f$ is $n-1$ continuous and

$$A(f) = \int_Q |Jf(x)| \, dx \, .$$

In this paper we prove the following

THEOREM. *If $f: Q \rightarrow R^n$ with $f^i \in W_{p_i}^1(Q)$, $p_i > n-1$ and $\sum_{i=1}^n 1/p_i \leqq 1$, then there is a nonnegative integer valued lower semicontinuous function $N(f, y)$ on $R^n$ such that*

$$A(f) = \int_{R^n} N(f, y) dy \tag{1}$$

*and, if* $\{f_j\}$ *is any sequence of quasi-linear mappings converging* $n-1$ *to* $f$ *with* $A(f) = \lim_{j\to\infty} a(f_j)$, *then*

$$\lim_{j\to\infty} \int_{R^n} | N(f, y) - N(f_j, y) | dy = 0 \tag{2}$$

*and*

$$\int_Q \phi(f(x))Jf(x)dx = \lim_{j\to\infty} \int_Q \phi(f_j(x))Jf_j(x)dx \tag{3}$$

*whenever* $\phi$ *is a continuous real valued function on* $R^n$ *with compact support.*

2. *Proof of* (1) *and* (2). Suppose $f$ satisfies the hypothesis of the theorem. By a full set of hyperplanes we will mean a subset $P$ of $\{P_i(r): 1 \leqq i \leqq n \text{ and } 0 < r < 1\}$ such that, for each $i$, $P_i(r) \in P$ for almost every $r \in I$.

If $\pi \subset Q$ is an $n$-cube such that $f \mid \partial\pi$ is continuous and $y \in R^n - f(\partial\pi)$, let $0(f, \pi, y)$ denote the topological index of $y$ with respect to the mapping $f \mid \partial\pi$ [4, p. 123]. If $y \in f(\partial\pi)$ let $0(f, \pi, y) = 0$.

Let $P$ be a full set of hyperplanes such that $f \mid P_i(r)$ is continuous whenever $P_i(r) \in P$. In harmony with [1, page 173] let, for $y \in R^n$,

$$N(f, y) = \sup \sum | 0(f, \pi, y) |$$

where the supremum is taken over all finite collections $G$ of non overlapping $n$-cubes $\pi \subset Q$ whose $n-1$ faces all lie in elements of $P$. From the properties of the topological index, it is easily seen that $N(f, y)$ is a lower semicontinuous function of $y$.

If $g: Q \to R^n$ is quasi-linear, then $N(g, y)$ is independent of the choice of $P$ and

$$a(g) = \int_{R^n} N(g, y)dy .$$

By [3, 3.5] we know that $f$ possesses a regular approximate differential almost everywhere in $Q$. Using the arguments of [1, page 424] one verifies that

$$\int_Q | Jf(x) | dx \leqq \int_{R^n} N(f, y)dy$$

whenever $N(f, y)$ is computed relative to a full set $P$ of hyperplanes such that the restriction of $f$ to each element of $P$ is continuous.

Suppose $\{f_j\}$ is a sequence of quasi-linear mappings converging $n-1$ to $f$ with $A(f) = \lim_{j\to\infty} a(f_j)$. Let $P$ be a full set of hyperplanes on each of which the sequence converges uniformly to $f$ and define $N(f, y)$ relative to $P$. For each $y \in R^n$ we have

$$N(f, y) \leqq \lim_{j\to\infty} N(f_j, y)$$

and hence

$$\int_{R^n} N(f, y)dy \leqq \lim_{j\to\infty} \int_{R^n} N(f_j, y)dy = A(f) .$$

If $\bar{P} \subset P$ is a full set of hyperplanes and $\bar{N}(f, y)$ is defined relative to $\bar{P}$, then, clearly $\bar{N}(f, y) \leqq N(f, y)$ for all $y \in R^n$. Since $A(f) = \int |Jf(x)| dx$, it follows that $N(f, y)$ is determined as an element of $L^1(R^n)$ independent of the choice of the sequence $\{f_j\}$. Thus (1) is proved and (2) follows because $N(f, y)$ is integer valued and

$$N(f, y) \leqq \lim_{j\to\infty} N(f_j, y)$$

for almost every $y \in R^n$ whenever $\{f_j\}$ is a sequence of quasilinear mappings converging $n-1$ to $f$ with $A(f) = \lim_{j\to\infty} a(f_j)$.

*Proof of* (3). Suppose $f$ and $\{f_j\}$ satisfy the conditions of the theorem and let $P$ be a full set of hyperplanes on each of which $\{f_j\}$ converges uniformly to $f$.

For $y \in R^n$ let

$$N^{\pm}(f, y) = \sup \sum_{\pi\in G} \frac{1}{2}[| 0(f, \pi, y) | \pm 0(f, \pi, y)]$$

where the supremum is taken over all finite collections $G$ of non overlapping $n$-cubes $\pi \subset Q$ whose $n-1$ faces all lie in elements of $P$. Clearly

$$N^{\pm}(f, y) \leqq N(f, y) \leqq N^{+}(f, y) + N^{-}(f, y) .$$

It is readily seen that

$$N^{\pm}(f, y) \leqq \lim_{j\to\infty} N^{\pm}(f_j, y)$$

and that the $N^{\pm}(f, y)$ are lower semicontinuous functions of $y$.

In case $g: Q \to R^n$ is quasi-linear, $N^{\pm}(g, y)$ are independent of the choice of $P$ and

$$N(g, y) = N^{+}(g, y) + N^{-}(g, y)$$

for almost every $y \in R^n$.

For each positive integer $j$, let

$$E_j^{\pm} = \{y: N^{\pm}(f_k, y) < N^{\pm}(f, y) \text{ for some } k \geqq j\} .$$

and let $E_j = E_j^{+} \cup E_j^{-}$.

Since the functions $N^{\pm}$ are integer valued we have

$$\lim_{j\to\infty} \mathscr{L}_n(E_j) = 0$$

where $\mathscr{L}_n$ denotes $n$ dimensional Lebesgue measure. Now

$$\begin{aligned}
&\int_{R^n} | N^+(f_j, y) - N^+(f, y) | dy \\
&\leqq \int_{R^n} N^+(f_j, y)dy - \int_{R^n - E_j^+} N^+(f, y)dy + \int_{E_j^+} (f, y)dy \\
&\leqq \int_{R^n} (N^+(f_j, y) + N^-(f_j, y))dy \\
&\quad - \int_{R^n - E_j} (N^+(f, y) + N^-(f, y))dy + \int_{E_j} N^+(f, y)dy \\
&\leqq \int_{R^n} N(f_j, y)dy - \int_{R^n - E_j} N(f, y)dy + \int_{E_j} N(f, y)dy \\
&= a(f_j) - A(f) + 2\int_{E_j} N(f, y)dy \,.
\end{aligned}$$

Thus

$$\lim_{j\to\infty} \int_{R^n} | N^\pm(f_j, y) - N^\pm(f, y) | dy = 0 \,.$$

Now

$$\begin{aligned}
0 &\leqq \int_{R^n} [N^+(f, y) + N^-(f, y) - N(f, y)]dy \\
&\leqq \int_{R^n} | N^+(f, y) - N^+(f_j, y) | dy + \int_{R^n} | N^-(f, y) - N^-(f_j, y) | dy \\
&\quad + \int_{R^n} | N(f, y) - N(f_j, y) | dy \,.
\end{aligned}$$

Thus, $N(f, y) = N^+(f, y) + N^-(f, y)$ for almost every $y \in R^n$.

Let $n(f, y) = N^+(f, y) - N^-(f, y)$. Then

$$\lim_{j\to\infty} \int_{R^n} | n(f, y) - n(f_j, y) | dy = 0 \,.$$

Suppose $\phi$ is a real valued continuous function on $R^n$ with compact support. If $g: Q \to R^n$ is quasi-linear (or of class $C^1$) then

$$\int_Q \phi(g(x))Jg(x)dx = \int_{R^n} \phi(y)n(g, y)dy \,.$$

Suppose $\{\bar{f}_j\}$ is a sequence of modifiers of $f$.

Then, from [3, 3.2], the sequence $\{\bar{f}_j\}$ converges $n - 1$ to $f$ and

$$\lim_{j\to\infty} \int_Q | Jf(x) - J\bar{f}_j(x) | dx = 0 \,.$$

Hence

$$\int_Q \phi(f(x))Jf(x)dx = \lim_{j\to\infty}\int_Q \phi(\bar{f}_j(x))J\bar{f}_j(x)dx$$
$$= \lim_{j\to\infty}\int_{R^n}\phi(y)n(\bar{f}_j, y)dy = \int_{R^n}\phi(y)n(f, y)dy .$$

Thus

$$\lim_{j\to\infty}\int_Q \phi(f_j(x))Jf_j(x)dx = \lim_{j\to\infty}\int_{R^n}\phi(y)n(f_j, y)dy$$
$$= \int_{R^n}\phi(y)n(f, y)dy = \int_Q \phi(f(x))Jf(x)dx$$

and (3) is proved.

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