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# A CONSTRUCTIVE RIEMANN MAPPING THEOREM*

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**Classically, the Riemann mapping theorem states that any open, simply connected and proper subset of $U$ of the complex plane is analytically equivalent to the open unit disk $S(0, 1)$. However this theorem is not constructively valid without some additional restriction on $U$. Two separate geometric conditions, mappability and maximal extensibility, on $U$ are then proposed. The two conditions are shown to be mathematically equivalent. Finally the mappability condition is shown to be both necessary and sufficient for an analytic equivalence to exist constructively between $U$ and $S(0, 1)$. The mappability condition is due Errett Bishop. The sufficiency proof is based on methods contained in [1].**

This paper is written from the constructive viewpoint that all mathematical statements should have a computational meaning. This viewpoint is developed in [1], and the background material in constructive analysis needed to read this paper can be found in the first five chapters of [1].

The intent here is to constructivize the Riemann mapping theorem, which is concerned with the question of when a set in the complex plane $\mathscr{C}$ has the same analytic structure as the open unit disk. Hence the following notion is basic to our study.

DEFINITION 1.1. Two open subsets $U_1$ and $U_2$ of $\mathscr{C}$ are *analytically equivalent* if there exist differentiable functions $f_1$: $U_1 \rightarrow U_2$ and $f_2$: $U_2 \rightarrow U_1$ such that $f_2 \circ f_1$: $U_1 \rightarrow U_1$ and $f_1 \circ f_2$: $U_2 \rightarrow U_2$ are the identity maps. $\square$

For the constructive definition of a differentiable function on an open set, see [1; p. 115]. The function $f_2$ is said to be the *inverse* to $f_1$. When there is no explicit need to mention $f_2$: $U_2 \rightarrow U_1$, we simply say $f_1$: $U_1 \rightarrow U_2$ is an *equivalence* of $U_1$ onto $U_2$. (The symbol $\square$ is used at the end of a definition or a proof, or at the end of the statement of a theorem or a corollary whose proof is not given.)

Under Definition 1.1, the classical Riemann mapping theorem states that any open, simply connected and proper subset $U$ of $\mathscr{C}$ is equivalent to $S(0, 1)$. (The notation $S(z, r) \equiv \{z': |z' - z| < r\}$ and $Sc(z, r) \equiv \{z': z' - z| \leqq r\}$ will be used.) This is not valid constructively without additional restrictions on $U$. The counter-example we have in mind is of a type introduced by Brouwer and later modified by Bishop in their critique of classical mathematics. Bishop defined the *limited* *principle of omniscience* to mean that given any sequence $\{n_k\}$ of the integers $\{0, 1\}$, either $n_k = 0$ for all $k$ or $n_k = 1$ for some $k$. Since there is no hope that a constructive proof of the limited principle of omniscience can ever be obtained, any hypothesis that implies the principle must be also nonconstructive. Here is our counter-example, based on a note in [1; p. 152]: If every open, simply connected and proper subset of $\mathscr{C}$ is equivalent to $S(0, 1)$, then the limited principle of omniscience holds. For a proof, let $\{n_k\}_{k=1}^{\infty}$ be a sequence of the integers $\{0, 1\}$. Define

$$U_k \equiv \begin{cases} S(0, 1) & \text{if} \quad n_k = 0 \\ S(0, 2) & \text{if} \quad n_k = 1 \end{cases}$$

and

$$U \equiv \bigcup_{k-1}^{\infty} U_k ,$$

then $U$ is clearly open, simply connected and a proper subset of $\mathscr{C}$. Suppose there exists an equivalence $f: S(0, 1) \rightarrow U$. Without loss in generality, assume $f(0) = 0$. Then either $|f'(0)| > 1$ or $|f'(0)| < 3/2$.

If $|f'(0)| > 1$, then we choose $\varepsilon$ and $r$ in $(0, 1)$ so that

$$(1) \qquad (1 + \varepsilon)r^{-1} < |f'(0)| .$$

Now we have

$$|f'(0)| = \left| (2\pi i)^{-1}\int_K f(z)z^{-2}dz \right| \leqq r^{-1}\|f\|_K ,$$

where $K \equiv \{z: |z| = r\}$. It follows from (1) that $1 + \varepsilon < \|f\|_K$. Therefore there exists $w \in U$ with $|w| > 1$. This means that $w \in U_k$ for some $k$. Hence $n_k = 1$.

If $|f'(0)| < 3/2$, then for a given $k$, suppose $n_k = 1$. Then $U = S(0, 2)$ and the mapping function has the property $|f'(0)| = 2$, which contradicts the assumption $|f'(0)| < 3/2$. Hence $n_k = 0$. (The reader will observe here that we have used the principle of the excluded middle in one of its *finite* forms: if $n_k = 1$ implies $0 = 1$, then $n_k = 0$. This is acceptable to the constructivist.) Therefore $n_k = 0$ for all $k$.

By assuming that $U$ is equivalent to $S(0, 1)$, we proved the limited principle of omniscience.

Therefore, a search for some additional restrictions on an open, simply connected and proper subset of $\mathscr{C}$ to assure its equivalence to $S(0, 1)$ is imperative if we wish to have a constructive Riemann mapping theorem.

Before we can state any additional restriction, some topological matters have to be discussed. It is well-known that both bounded and unbounded sets can be equivalent to $S(0, 1)$. However, it is awkward constructively to separate the two cases. Therefore we will often use, instead of the ordinary metric $\rho(z, z') \equiv |z - z'|$, the bounded metric $d$ defined by

$$d(z, z') \equiv 2|z - z'|(1 + |z|^2)^{-1/2}(1 + |z'|^2)^{-1/2}$$

for all $z, z' \in \mathscr{C}$. See [2; p. 43] for a proof that $d$ is indeed a metric. This metric will also be used in a more general context. Let $F$ be the family of all totally bounded subsets of $\mathscr{C}$ relative to $d$. Then

$$d(z, B) \equiv \inf \{d(z, z') : z' \in B\}$$

exists for each $z \in \mathscr{C}$ and each $B \in F$. Also

$$d^*(A, B) \equiv \inf \{d(z, B) : z \in A\}$$

is properly defined on $F \times F$. In contrast to the metric complement defined in [1; p. 83], we define the *complement* $\sim A$ of an arbitrary set $A \subset \mathscr{C}$ relative to $d$ to be the set

$$\sim A \equiv \{z : d(z, z') > 0 \text{ whenever } z' \in A\} .$$

We are now ready to introduce a very important notion.

DEFINITION 1.2. A nonvoid, open and simply connected subset $U$ of $\mathscr{C}$ is *mappable* if there exists a *distinguished point* $z_0$ in $U$ such that for each $\varepsilon > 0$ there exists a subfinite set $B \equiv \{z_1, \cdots, z_n\}$ in $\sim U$ such that any path $\gamma$, with left endpoint $z_0$ and $d^*(\gamma, B) \geqq \varepsilon$, lies in $U$. $B$ is called an *$\varepsilon$-border* of $U$ relative to $z_0$. $\square$

The above definition is due to Errett Bishop. It is intended to replace his earlier definition [1; p. 145] of a mappable set $U$, which turned out to be only a sufficient condition for the existence of an equivalence of $U$ with $S(0, 1)$. For an example to show that the earlier definition is not a necessary condition, construct the sequence $\{a_k : k \geqq 1\}$ of integers such that $a_k = 0$ if $2k + 2$ is the sum of two positive primes less than $2k + 2$ and $a_k = 1$ if it is not. Define

$$A_k \equiv \begin{cases} \{z : |z| < 1\} \text{ if } a_k = 0 \\ \{z : |z - 2| < 1 + n^{-1}\} \text{ if } a_k = 1 \end{cases}$$

and

$$U \equiv \bigcup_{k=1}^{\infty} A_k .$$

The set $U$ cannot be shown to satisfy the earlier definition of mappability, although it is equivalent to $S(0, 1)$. In fact the set $U$, with

0 as its distinguished point, is mappable according to Definition 1.2 and we will show that an open set is equivalent to $S(0, 1)$ if and only if it is mappable. Hence this definition supercedes the earlier one.

Sometimes when we want to emphasize the distinguished point $z_0$, we will write $(U, z_0)$ for the mappable set $U$. Note that an $\varepsilon$-border acts only as an approximate boundary and does not directly require $U$ to have a nonvoid boundary. However, we can define in a sense a distance from any point $z$ in a mappable set $U$ to its complement $\sim U$. It is convenient at this point to have a notation for a neighborhood about a point $z \in \mathscr{C}$ relative to the metric $d$. Let

$$D(z, r) \equiv \{z' : d(z, z') < r\}$$

and

$$Dc(z, r) \equiv \{z' : d(z, z') \leqq r\} .$$

Also define

$$A \sim B \equiv A \cap (\sim B)$$

for all subsets $A, B$ of $\mathscr{C}$.

DEFINITION 1.3. A nonvoid, open and simply connected subset $U$ of $\mathscr{C}$ is said to have the *maximal extent property* if for each $z \in U$ there exists a real number $\mu > 0$ such that $D(z, \mu) \subset U$ and for each $\mu' > \mu$ there exists $z' \in D(z, \mu') \sim U$. $\mu$ is called the *maximal extent* of $U$ about $z$. $\square$

Note that we obtained the counter-example to the classical Riemann mapping theorem by exhibiting a set $U$ for which we don't know the maximal extent of $U$ about any point $z \in U$. In fact, Definition 1.3 contains precisely the type of additional restriction on $U$ that we need, because the following statements are mathematically equivalent:

(1) $U$ is analytically equivalent to $S(0, 1)$,

(2) $U$ is mappable,

(3) $U$ has the maximal extent property.

In § II, we prove that conditions (2) and (3) are equivalent. In § III, we prove that conditions (1) and (2) are equivalent. Classically any open, simply connected and proper subset of $\mathscr{C}$ trivially has the maximal extent property and hence is mappable. Thus the equivalence of (1) and (2) may be regarded as a constructive substitute for the classical Riemann mapping theorem and its converse.

**II. Mappable sets.** Note that mappability is a global condition on a set $U$, whereas maximal extensibility is a local condition on $U$.

Nonetheless, these conditions are equivalent, as we will now demonstrate in two steps. Only Proposition 2.1 will be used later.

PROPOSITION 2.1. *Any mappable set has the maximal extent property.*

*Proof*. Let $(U, z_0)$ be a mappable set. First we show that $U$ has a maximal extent about its distinguished point $z_0$. For each $\varepsilon > 0$, define $\theta(\varepsilon) \equiv d(z_0, B(\varepsilon))$, where $B(\varepsilon)$ is any $\varepsilon$-border of $U$ relative to $z_0$. Although $\theta$ is only an operation on $(0, \infty)$ we do have the following essential inequality:

$$|\theta(\varepsilon) - \theta(\delta)| \leqq \max\{\varepsilon, \delta\} \tag{1}$$

for all $\varepsilon, \delta \in (0, \infty)$. Because of symmetry, we prove (1) when we prove the inequality

$$\theta(\delta) \leqq \theta(\varepsilon) + \max\{\varepsilon, \delta\}\ . \tag{2}$$

Suppose $\theta(\delta) > \theta(\varepsilon) + \max\{\varepsilon, \delta\}$. Then there exists $z \in B(\varepsilon)$ such that $\theta(\delta) > d(z_0, z) + \delta$. Hence there exists a path $\gamma$ with left endpoint $z_0$ and right endpoint $z$ such that $\gamma \subset D(z_0, \theta(\delta) - \delta)$. By the definition of $\theta(\delta)$, $\gamma \subset U$ and hence $z \in U$. But $z \in B(\varepsilon) \subset \sim U$. This contradiction implies (2).

Using the inequality (1), one can easily show that the limit $\mu \equiv \lim_{\varepsilon \to 0} \theta(\varepsilon)$ exists and is unique. Moreover

$$\mu \leqq \theta(\varepsilon) \tag{3}$$

for each $\varepsilon > 0$ and each $\varepsilon$-border $B(\varepsilon)$. It follows from (3) that $D(z_0, \mu) \subset U$. Now for each $\mu' > \mu$, we can choose $z' \in B(2^{-1}(\mu' - \mu))$ so that $z' \in D(z_0, \mu') \sim U$. Hence $\mu$ is the maximal extent of $U$ about $z_0$.

To show that $U$ has a maximal extent about any other point $z \in U$, connect the distinguished point $z_0$ to $z$ by a path $\gamma'$ in $U$ such that $\gamma'$ has left endpoint $z_0$ and right endpoint $z$. Then

$$\gamma'_{2r} \equiv \{z : d(z, \gamma') \leqq 2r\} \subset U \tag{4}$$

for some $r > 0$. We will use (4) to show that $z$ can also serve as the distinguished point of $U$. Let $\varepsilon > 0$. Define $\delta \equiv \min\{\varepsilon, r\}$. Let $B$ be a $\delta$-border of $U$ relative to $z_0$. Let $\gamma$ be any path with left endpoint $z$ and

$$d^*(\gamma, B) \geqq \varepsilon\ . \tag{5}$$

Then $d^*(\gamma' + \gamma, B) \geqq \delta$ because of conditions (4) and (5). Since $\gamma' + \gamma$ is a path with left endpoint $z_0$, we conclude $\gamma' + \gamma \subset\subset U$ or $\gamma \subset\subset U$. Therefore $B$ can serve as an $\varepsilon$-border of $U$ relative to $z$.

Since $U$ has a maximal extent about its distinguished point, it has a maximal extent about every point in $U$. □

Intuitively, one feels that in knowing the distance of any point $z$ in a set $U$ to the complement $\sim U$, one also knows approximately both the shape and size of $U$. This idea is formulated as the converse of Proposition 2.1.

PROPOSITION 2.2. *Any set with the maximal extent property is mappable.*

*Proof.* Let $U$ be a set with the maximal extent property. Since $U$ is nonvoid, choose any point $z_0 \in U$ and let it be the distinguished point in $U$. We intend to find an $\varepsilon$-border of $U$ relative to $z_0$, for each $\varepsilon > 0$.

For a given $\delta > 0$, observe that there is an integer $n \geqq 1$ such that for any disk $D(z, r)$ with $r > 0$ there exist points $z_1, \cdots, z_n$ in $D(z, r)$ such that $d(z, \{z_1, \cdots, z_n\}) \geqq r - \delta$ and such that any path $\gamma$ in $\mathscr{C}$, with left endpoint $z$ and $d(\gamma, \{z_1, \cdots, z_n\}) \geqq \delta$, lies in $D(z, r)$. We say $\{z_1, \cdots, z_n\}$ is a $\delta$-*net* of $D(z, r)$. Each point $z_k$ of the $\delta$-net is said to be *generated* from $z$ and this relationship is denoted by $z < z_k$. The reason for introducing the concept of $\delta$-nets is simple. The only way we have of getting an $\varepsilon$-border, which consists of points in the complement $\sim U$, is to find points in $U$ such that the maximal extents of $U$ about these points are small.

Fix $\varepsilon > 0$. For each point in $U$, let $\mu(z)$ be the maximal extent of $U$ about $z$. Now we construct recursively a set $P$ of points in $U$. First, place the distinguished point $z_0$ in $P$. Second, if $z \in P$, then place one and only one $2^{-5}\varepsilon$-net of $D(z, \mu(z))$ into $P$. In general $P$ is a countable set. We extract a subfinite set $Q$ from $P$ as follows. Let $z_0 \in Q$. For each finite sequence $z_0 < z_1 < \cdots < z_n$ of points in $P$, $z_i$ generated from $z_{i-1}$, $z_0$ the distinguished point in $U$ and

$$\{z_0, \cdots, z_{n-1}\} \subset Q ,$$

define

$$\alpha \equiv \min \{d(z_i, z_j) : 1 \leqq i, j \leqq n, i \neq j\} . \tag{1}$$

Either $\alpha > 2^{-6}\varepsilon$ or $\alpha < 2^{-5}\varepsilon$. If $\alpha > 2^{-6}\varepsilon$, then place $z_n$ in $Q$. If $\alpha < 2^{-5}\varepsilon$, then discard $z_n$. Since the metric space $\mathscr{C}$ relative to $d$ is totally bounded, we see that $Q$ is subfinite. For each $z \in Q$, choose a point

$$\zeta(z) \in D(z, \mu(z) + 2^{-4}\varepsilon) \sim U . \tag{2}$$

This is always possible because $U$ has the maximal extent property. Then the set

$$B \equiv \{\zeta(z)\colon z \in Q\} \tag{3}$$

is an $\varepsilon$-border of $U$ relative to $z_0$. To prove this, let $\gamma$ be any path with left endpoint $z_0$ and

$$d^*(\gamma, B) \geqq \varepsilon \,. \tag{4}$$

We will show that

$$\{z\colon d(z, \gamma) \leqq 2^{-4}\varepsilon\} \subset U \tag{5}$$

and hence $\gamma$ lies in $U$. The method is to divide $\gamma$ into a finite partition of subarcs, each of which is contained in a disk that is shown to be well contained in $U$. Let $[0, 1]$ be the parameter interval of $\gamma$. Since $\gamma$ is piecewise differentiable, then there exists an integer $m \geqq 1$ and points $0 = t_0 < \cdots < t_m = 1$ such that

$$\gamma([t_i, t_{i+1}]) \subset Dc(w_i, 2d(w_i, w_{i+1})) \,, \tag{6}$$

where $\gamma(t_i) = w_i$ $(0 \leqq i \leqq m)$, and

$$2d(w_i, w_{i+1}) \leqq 2^{-4}\varepsilon \qquad (0 \leqq i \leqq m-1) \,. \tag{7}$$

Now define

$$D_i = Dc(w_i, 2d(w_i, w_{i+1}) + 2^{-4}\varepsilon) \qquad (0 \leqq i \leqq m-1) \,. \tag{8}$$

By induction on the integer $i$, we will prove that there exists $z_i \in Q$ for each $i \in \{0, \cdots, m-1\}$ such that

$$D_i \subset D(z_i, \mu(z_i)) \subset U \,. \tag{9}$$

Without loss in generality, assume $\mu(z_0) > 2^{-2}\varepsilon$. Otherwise $\mu(z_0) < 2^{-1}\varepsilon$ and an $\varepsilon$-border of $U$ consists of a single point in $D(z_0, 2^{-1}(\mu(z_0) + 2^{-1}\varepsilon)) \sim U$. Assuming $\mu(z_0) > 2^{-2}\varepsilon$, we have

$$D_0 = Dc(z_0, 2d(z_0, w_1) + 2^{-4}\varepsilon) \subset Dc(z_0, 2^{-3}\varepsilon) \subset D(z_0, \mu(z_0))$$

by conditions (7) and (8). Now suppose (9) is verified for all $i \leqq k$, with $z_1, \cdots, z_k \in Q$. Either

$$d(w_{k+1}, z_k) < \mu(z_k) - 2^{-3}\varepsilon \tag{10}$$

or

$$d(w_{k+1}, z_k) > \mu(z_k) - 3 \cdot 2^{-4}\varepsilon \,. \tag{11}$$

If (10) holds, then

$$D_{k+1} \subset Dc(w_{k+1}, 2^{-3}\varepsilon) \subset Dc(z_k, 2^{-3}\varepsilon + d(w_{k+1}, z_k)) \subset D(z_k, \mu(z_k)) \,.$$

In this case, we just choose $z_{k+1} = z_k$. If (11) holds, then there exists $z \in P$ such that $z$ is contained in a $2^{-4}\varepsilon$-net of $z_k$ and

$$d(z, w_{k+1}) \leqq 2^{-2}\varepsilon \ . \tag{12}$$

By definition of the set $Q$, there exists $z' \in Q$ such that

$$d(z, z') < 2^{-4}\varepsilon \ . \tag{13}$$

By conditions (2), (3), and (4) there exists $\zeta(z') \in D(z', \mu(z') + 2^{-4}\varepsilon) \sim U$ such that

$$d(w_{k+1}, \zeta(z')) \geqq d^*(\gamma, B) \geqq \varepsilon \ . \tag{14}$$

In view of (12), (13), and (14), we compute

$$\begin{aligned} \mu(z') &\geqq d(\zeta(z'), z') - 2^{-4}\varepsilon \\ &\geqq d(\zeta(z'), w_{k+1}) - d(w_{k+1}, z') - 2^{-4}\varepsilon \\ &\geqq d(\zeta(z'), w_{k+1}) - d(w_{k+1}, z) - d(z, z') - 2^{-4}\varepsilon \\ &\geqq \varepsilon - 2^{-2}\varepsilon - 2^{-4}\varepsilon - 2^{-4}\varepsilon \\ &> 2^{-1}\varepsilon \ . \end{aligned} \tag{15}$$

However

$$\begin{aligned} &d(z', z) + d(z, w_{k+1}) + 2d(w_{k+1}, w_{k+2}) + 2^{-4}\varepsilon \\ &\quad \leqq 2^{-4}\varepsilon + 2^{-2}\varepsilon + 2^{-4}\varepsilon + 2^{-4}\varepsilon \\ &\quad < 2^{-1}\varepsilon \ . \end{aligned} \tag{16}$$

The inequalities (15) and (16) together imply

$$D_{k+1} \subset D(z', \mu(z')) \ .$$

Hence we choose $z_{k+1} \equiv z'$. This completes the induction to construct $\{z_i\colon 0 \leqq i \leqq m-1\}$ that satisfy (9).

In view of (6), condition (9) implies (5). Hence $\gamma \subset U$. The upshot is that $B$ is an $\varepsilon$-border of $U$ relative to $z_0$. Hence $U$ is a mappable set with distinguished point $z_0$. □

**III. The Riemann mapping theorem and its converse.** Although the ordinary metric $\rho$ and the bounded metric $d$ on $\mathscr{C}$ are not equivalent metrics, they do share some important topological properties. A subset $U$ of $\mathscr{C}$ is open relative to $\rho$ if and only if it is open relative to $d$. A subset $K$ of an open set $U$ is compact and well contained in $U$ relative to $\rho$ if and only if it is compact and well contained in $U$ relative to $d$. Therefore two open subsets of $\mathscr{C}$ are (analytically) equivalent if they are equivalent relative to either metric. We will continue to use both metrics on $\mathscr{C}$ because many results that we will use are stated in terms of the metric $\rho$, whereas mappable sets are best described in terms of the metric $d$. However, if we stipulate that a mappable set $U$ is bounded relative to $\rho$, then the concepts of $\varepsilon$-borders and maximal extents of $U$, defined originally in terms of $d$, can be and will be accepted as defined in terms of $\rho$. Also the results of §II, expressed in terms of $d$, will continue to hold under the metric $\rho$.

It is our intention here to prove a constructive version of the Riemann mapping theorem, that is, every mappable set $U$ is equivalent to $S(0, 1)$. The method of constructing the mapping function from $U$ to $(0, 1)$ was invented by Koebe and later modified by Ostrowski [3]. Our proof will follow closely the development given by Bishop [1] to the ideas of Koebe and Ostrowski.

DEFINITION 3.1. A mappable set $U$ is *sequestered* if $U \subset S(0, 1)$ and for each $\varepsilon > 0$ there exists an $\varepsilon$-border $K$ of $U$ such that $K \subset S(0, 1)$. □

LEMMA 3.2. *Let $U$ be a sequestered set with distinguished point $z$ and suppose $0 \in S(0, 1) \sim U$. Let $s$ be any branch of the square root function on $U$: $s(\zeta) \equiv \exp(2^{-1} \log \zeta)$. Then $U_0 \equiv s(U)$ is a sequestered set with distinguished point $s(z)$ and $s\colon U \to U_0$ is an equivalence.*

*Proof.* The map $s_0$ defined by $s_0(w) \equiv w^2$ of $U_0$ onto $U$ is inverse to $s$. Therefore $U$ is equivalent to $U_0$.

To show that $U_0$ is mappable, fix $\varepsilon > 0$ and choose an $\varepsilon^2$-border $K$ of $U$ such that $K \subset S(0, 1)$. Let $K_0 \equiv \{w\colon w^2 \in K\}$. Then $K_0 \subset \sim U_0$. Let $w \in K_0$ and $w' \in U_0$. We want to show that $w \neq w'$. Since $U_0$ is open, let $S(w', r) \subset U_0$ for some $r > 0$. Suppose $|w - w'| < r$. Then $w \in U_0$ and there exists $z \in U$ such that $z = w^2$. But $w^2 \in K \subset \sim U$. This contradiction implies $|w - w'| \geqq r$. Hence $K_0 \subset \sim U_0$. Also, for any totally bounded set $L_0 \subset S(0, 1) \sim K_0$,

$$(1) \qquad [\rho^*(L_0, K_0)]^2 \leqq \rho^*(s_0(L_0), K) .$$

The above inequality is proved in [1; p. 146]. Since $K$ is an $\varepsilon^2$-border of $U$, (1) implies $K_0$ is an $\varepsilon$-border of $U_0$ relative to $z_0 \equiv s(z)$. Thus $(U_0, z_0)$ is a mappable set. Since $U_0 \subset S(0, 1)$ and $\varepsilon$ is an arbitrary positive number, it follows that $U_0$ is sequestered. □

DEFINITION 3.3. Let $(U, 0)$ be a sequestered set and let $\mu$ be the maximal extent of $U$ about 0. Choose $a \in S(0, 1) \sim U$ and $\alpha$ with $|\alpha| = 1$ such that

$$(1) \qquad \mu \leqq |a| \leqq 2^{-1}(1 + \mu) \quad \text{and} \quad \alpha a < 0 .$$

For each $\beta \in \mathscr{C}$ with $|\beta| < 1$ define the function $h_\beta\colon S(0, 1) \to S(0, 1)$ by

$$h_\beta(z) \equiv (z - \beta)(1 - \beta^* z)^{-1}, \tag{2}$$

where $\beta^*$ is the conjugate of $\beta$. Then the the function

$$\phi_U \equiv \alpha^* h_b \circ s \circ \alpha h_a \tag{3}$$

defined on $U$ with $b \equiv |a|^{1/2}$, is called the *canonical map* of $U$. $\square$

LEMMA 3.4. *Let $(U, 0)$ be a sequestered set and let $\mu$ be the maximal extent of $U$ about 0. Then the canonical map $\phi_U$ is an equivalence of $(U, 0)$ onto a sequestered set $(U^*, 0)$ such that*

(i) $\phi_U(0) = 0$,

(ii) $\phi'_U(0) > 1 + \dfrac{1}{32}(1 - \mu)^2$

*and*

(iii) $\mu \leqq \mu^*$,

*where $\mu^*$ is the maximal extent of $U^*$ about 0.*

*Proof.* By (2) of Definition 3.3, we see that $\alpha h_a$ is an equivalence of $U$ with a set $U_0$ such that $0 \in S(0, 1) \sim U_0$. By Lemma 3.2, we see that $s$ is an equivalence of $U_0$ with $s(U_0)$. Using (2) of Definition 3.3 again, we show that $\alpha^* h_b$ is an equivalence of $s(U_0)$ with a sequestered set $U^*$. Hence $\phi_U$ is an equivalence of $U$ with $U^*$ such that $\phi_U(0) = 0$. By the chain rule for differentiation,

$$\phi'_U(0) = 2^{-1}|a|^{-1/2}(1 + |a|). \tag{1}$$

By condition (1) of Definition 3.3,

$$\begin{aligned} 2^{-1}|a|^{-1/2}(1 + |a|) &= 1 + 2^{-1}|a|^{-1/2}(1 - |a|^{1/2})^2 \\ &> 1 + 2^{-1}(1 - |a|^{1/2})^2 \\ &\geqq 1 + 2^{-1}(1 - (2^{-1}(1 + \mu))^{1/2})^2 \\ &\geqq 1 + 2^{-1}\left(1 - \left(\frac{3}{4} + \left(\frac{1}{4}\right)\mu\right)\right)^2 \\ &= 1 + \frac{1}{32}(1 - \mu)^2. \end{aligned} \tag{2}$$

Combining (1) and (2), we have conclusion (ii).

To show that $\mu \leqq \mu^*$, we observe that the inverse $\psi\colon U^* \to U$ of $\phi_U$ is the composition of the map $z \to h_{-b}(\alpha z)$, the map $z \to z^2$, and the map $z \to h_{-a}(\alpha^*)$. Each of these maps is a function from $S(0, 1)$ into itself. Hence $|\psi(z)| \leqq |z|$ whenever $|z| < 1$, by the Schwarz lemma. Moreover, we have

$$(3)\qquad \begin{aligned}|h_a(z) - h_a(z')| &= \frac{|(1 - aa^*)(z - z')|}{|(1 - a^*z)(1 - a^*z')|} \\ &\leqq \frac{1 - |a|^2}{(1 - |a|)^2}|z - z'| \leqq \left(\frac{1 + |a|}{1 - |a|}\right)|z - z'| ,\end{aligned}$$

for all $z, z' \in S(0, 1)$. Now let $B$ be a $\delta$-border of $U$ relative to 0, where

$$(4)\qquad \delta \equiv \varepsilon^2\left(\frac{1 + |b|}{1 - |b|}\right)^{-2}\left(\frac{1 + |a|}{1 - |a|}\right)^{-1} .$$

Let $C \equiv \alpha h_a(B)$, $C^* \equiv \{w: w^2 \in C\}$ and $B^* \equiv a^*h_b(C^*)$. Then $B^*$ is an $\varepsilon$-border of $U^*$ by Lemma 3.2, equation (3) of Definition 3.3, (3) and (4). Since $\psi(B^*) = B$ and $|\psi(z)| \leqq |z|$ whenever $|z| < 1$, we conclude that

$$(5)\qquad \rho(0, B) \leqq \rho(0, B^*) .$$

Letting $\varepsilon \to 0$, it follows from (5) that $\mu \leqq \mu^*$. This proves (iii). □

LEMMA 3.5. *Let* $(U_0, 0)$ *be a sequestered set and* $\phi_0$ *be the canonical map of* $(U_0, 0)$ *onto* $(U_1, 0) \equiv (U_0^*, 0)$. *Continuing in this way, we define a sequence* $\{(U_n, 0)\}_{n=0}^\infty$ *of sequestered sets and a sequence of canonical maps* $\{\phi_n: (U_n, 0) \to (U_{n+1}, 0)\}_{n=0}^\infty$. *Let* $\mu_n$ *be the maximal extent of* $U_n$ *about* 0. *Then*

(i) $\mu_n \leqq \mu_{n+1} \qquad (n \geqq 0)$

*and*

(ii) $1 + \frac{1}{32}(1 - \mu_n)^2 < \mu_0^{-1/(n+1)} \qquad (n \geqq 1)$ .

*Proof*. By (iii) of Lemma 3.4, we have $\mu_n \leqq \mu_{n+1}$ whenever $n \geqq 0$. For each $n \geqq 1$, write

$$\phi_0^{n+1} \equiv \phi_n \circ \cdots \circ \phi_0: (U_0, 0) \longrightarrow (U_{n+1}, 0) .$$

Also write $Df \equiv f'$ for any differentiable function $f$. By Corollary 3 of Theorem 5 of Chapter 5 of [1], $|(D\phi_0^{n+1})(0)| \leqq r^{-1}$, whenever $r < \mu_0$. Therefore

$$(1)\qquad |(D\phi_0^{n+1})(0)| \leqq \mu_0^{-1} .$$

On the other hand,

$$\begin{aligned}|(D\phi_0^{n+1})(0)| &= |\phi_n'(0) \cdots \phi_0'(0)| \\ &\geqq \left(1 + \frac{1}{32}(1 - \mu_n)^2\right) \cdots \left(1 + \frac{1}{32}(1 - \mu_0)^2\right) ,\end{aligned}$$

by (ii) of Lemma 3.4. Hence

$$(2) \qquad |(D\phi_0^{n+1})(0)| \geqq \left(1 + \frac{1}{32}(1-\mu_n)^2\right)^{n+1} .$$

Inequalities (1) and (2) together imply (ii). □

Note that (ii) of Lemma 3.5 implies $\mu_n \to 1$ as $n \to \infty$. We can now prove the Riemann mapping theorem for sequestered sets.

LEMMA 3.6. *For each sequestered set* $(U_0, 0)$ *the maps*

$$\phi_0^n \equiv \phi_{n-1} \circ \cdots \circ \phi_0 \colon (U_0, 0) \longrightarrow (U_n, 0)$$

*converge uniformly on compact subsets well contained in* $U_0$ *to an equivalence* $\phi$ *of* $(U_0, 0)$ *with* $S(0, 1)$.

*Proof.* For each $m < n$ define

$$\phi_m^n \equiv \phi_{n-1} \circ \cdots \circ \phi_m \colon U_m \longrightarrow U_n .$$

Let $L \subset\subset U_0$ be a compact set. By the Corollary to Proposition 7 of Chapter 5 of [1], there exists $c < 1$ such that $|\phi_0^n(z)| \leqq c$ for all $n \geqq 1$ and $z \in L$. Let $\varepsilon > 0$ be arbitrarily chosen. Then we find $R \in (0, 1)$ such that

$$(1) \qquad R^2 - c > (1 - R)^{1/4}$$

and

$$(2) \qquad 3(1 - R)^{1/4} \leqq \varepsilon .$$

Because of (ii) of Lemma 3.5, there exists $N \geqq 1$ so that $\mu_n \geqq R$ whenever $n \geqq N$. Therefore $S(0, R) \subset U_n$ for all $n \geqq N$. For each $m < n$, let $\psi_n^m \colon U_n \to U_m$ be the inverse to $\phi_m^n$. If $m, n \geqq N$, then $S(0, R) \subset U_m \cap U_n$ and $U_m \cup U_n \subset S(0, 1)$. By the Corollary to Proposition 8 of Chapter 5 of [1],

$$|\phi_m^n(\zeta) - \zeta| \leqq 3(R^2 - r)^{-1}(1 - R)^{1/2}$$

whenever $r \equiv |\zeta| < R^2$. Therefore, for $n > m \geqq N$,

$$\begin{aligned} |\phi_0^n(z) - \phi_0^m(z)| &= |\phi_m^n(\phi_0^m(z)) - \phi_0^m(z)| \\ &\leqq \varepsilon \end{aligned}$$

whenever $z \in L$, in view of conditions (1) and (2). Since $\varepsilon$ is arbitrarily chosen, it follows that $\{\phi_0^n\}$ converges uniformly on $L$, to a continuous function from $L$ to $Sc(0, c)$. Since $L$ is an arbitrary compact set with $L \subset\subset U_0$, the sequence $\{\phi_0^n\}$ converges on $U_0$ to a differentiable function $\phi \colon U_0 \to S(0, 1)$.

To construct the inverse $\psi \colon S(0, 1) \to U_0$ of $\phi$, consider a compact set $J \subset\subset S(0, 1)$. For each $m < n$, recall that $\psi_n^m \colon U_n \to U_m$ is the inverse to $\phi_m^n$. Since $\mu_n \to 1$ as $n \to \infty$, the functions $\psi_n^0$ are defined on $J$ for all sufficiently large $n$. Moreover, there exists a compact set $K \subset\subset U$ and an integer $N \geqq 1$ such that $\phi_0^n(K) \supset J$ whenever $n \geqq N$. Using the Corollary to Proposition 8 of Chapter 5 of [1] again, we see that $\phi_m^n$ converges uniformly on $K$ to the identity function $z \to z$ as $m, n \to \infty$. It follows that for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi_0^n(z) - \phi_0^n(z')| > \delta$ whenever $n \geqq N$, $z \in K$, $z' \in K$ and $|z - z'| > \varepsilon$. Therefore $|\psi_n^0(\zeta) - \psi_n^0(\zeta')| \leqq \varepsilon$ whenever $n \geqq N$, $\zeta \in J$, $\zeta' \in J$ and $|\zeta - \zeta'| \leqq \delta$. Hence the sequence $\{\psi_n^0\}$ converges on $J$ to a differentiable map $\psi: J \to U_0$. Since $J$ is an arbitrary compact set with $J \subset\subset S(0, 1)$, the function $\psi$ may be extended to a differentiable function $\psi: S(0, 1) \to U_0$ on the entire unit disk $S(0, 1)$.

To show that $\phi: U_0 \to S(0, 1)$ is an equivalence, it remains to show that $\psi \circ \phi: U_0 \to U_0$ and $\phi \circ \psi: S(0, 1) \to S(0, 1)$ are the identity maps. For each $z \in U_0$, the points $\{\phi_0^n(z)\}$ lie in some compact set $J \subset\subset S(0, 1)$. Now $\psi_k^0$ is defined on $J$ for $k$ sufficiently large. Therefore, for $\varepsilon > 0$,

$$|\psi_k^0(\phi_0^n(z)) - \psi(\phi_0^n(z))| \leqq \varepsilon$$

whenever $k$ is sufficiently large. Taking $n = k$, we have

$$|z - \psi(\phi_0^n(z))| \leqq \varepsilon \ .$$

Now let $n \to \infty$. Then $|z - \psi(\phi(z))| \leqq \varepsilon$. Since $\varepsilon$ is arbitrary, $\psi \circ \phi$ is the identity map on $U_0$. Similarly, we show that $\phi \circ \psi$ is the identity map on $S(0, 1)$. □

THEOREM 3.7. *Every mappable set is equivalent to* $S(0, 1)$.

*Proof*. Let $U$ be a mappable set with distinguished point $\alpha$. Because of Lemma 3.6, it suffices to show that $(U, \alpha)$ is equivalent to a sequestered set $(V, 0)$. Since an $\varepsilon$-border of $U$ is a nonvoid set, there exists $\zeta \in \sim U$. Without loss in generality, assume $\zeta = 0$. Since $U$ is simply connected, there exists a branch of the square root function $s$ defined by $s(z) \equiv \exp(2^{-1} \log z)$ that is an equivalence of $U$ with some subset $U_0$ of $\mathscr{C}$. The map $s_0$ defined by $s_0(w) \equiv w^2$ from $U_0$ onto $U$ is inverse to $s$. We claim that $U_0$ is mappable. Let $\alpha_0 \equiv s(\alpha)$ be the distinguished point in $U_0$. Let $K$ be a $2^{-1}\varepsilon^2$-border of $U$ relative to $\alpha$. Let

$$K_0 \equiv \{w: w^2 \in K\} \ .$$

Consider any totally bounded set $L_0 \subset \sim K_0$. Then

$$d^*(s_0(L_0), K) = \inf\left\{\frac{2|z^2 - w|}{\sqrt{1 + |z|^4}\sqrt{1 + |w|^2}}: z \in L_0, w \in K\right\}$$

$$
\begin{aligned}
&= \inf\left\{\frac{2|z^2 - w^2|}{\sqrt{1+|z|^4}\sqrt{1+|w|^4}} : z \in L_0,\ w \in K_0\right\} \\
(1) \qquad &\geqq 2^{-1} \inf\left\{\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}\right. \\
&\qquad \left. \cdot \frac{2|z+w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} : z \in L_0,\ w \in K_0\right\} \\
&= 2^{-1}\left[\inf\left\{\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} : z \in L_0,\ w \in K_0\right\}\right]^2 \\
&= 2^{-1}[d^*(L_0, K_0)]^2 .
\end{aligned}
$$

Now suppose $\gamma_0$ is a path such that $\gamma_0(0) = \alpha_0$ and $d^*(\gamma_0, K_0) \geqq \varepsilon$. Then (1) implies $d^*(s_0(\gamma_0), K) \geqq 2^{-1}\varepsilon^2$. Since $s_0(\gamma_0(0)) = \alpha$ and $K$ is a $2^{-1}\varepsilon^2$-border of $U$, we see that $s_0(\gamma_0) \subset\subset U$. Hence $\gamma_0 \subset\subset U_0$. Therefore $K_0$ is an $\varepsilon$-border of $U_0$ relative to $\alpha_0$. Therefore $U_0$ is mappable.

Note that $U_0 \equiv s(U)$ is a nonvoid open set. Therefore $U_0$ contains an open sphere $S(a, r)$ with

$$(2) \qquad 0 < r < |a| .$$

Then we claim that

$$(3) \qquad S(-a, 2^{-1}r) \subset \sim U_0 .$$

To prove (3), choose an arbitrary point $w \in S(-a, 2^{-1}r)$ and an arbitrary point $w' \in U_0$. Suppose $|w - w'| < 2^{-1}r$. Then $w' \in S(-a, r)$. Since $S(a, r) \subset U_0$, there exists $w'' \in U_0$ such that $w'' = -w'$. Then $S_0(w'') = (w'')^2 = (w')^2 = s_0(w')$ implies $w'' = s \circ s_0(w'') = s \circ s_0(w') = w'$. However $w'' \neq w'$ according to (2). This contradiction gives $|w - w'| \geqq 2^{-1}r$. Hence we conclude (3).

Then the function defined by

$$t(z) \equiv r[(s(z) + a)^{-1} - (s(\alpha) + a)^{-1}]$$

is an equivalence of $U$ with some sequestered set $(V, 0)$. □

So we have exhibited a family of sets (the mappable sets) that are equivalent to $S(0, 1)$. But is this family exhaustive in the sense that every open set equivalent to $S(0, 1)$ is mappable? To answer this question in the affirmative, we need several lemmas. The first one is the famous Koebe covering theorem, which has a classical proof that is essentially constructive, as given for instance in [4; p. 276]. The second lemma has a simple proof, which will also be omitted.

An equivalence $f: S(0, 1) \to U$ is *normalized* if $f(0) = 0$ and $f'(0) = 1$.

LEMMA 3.8. (*Koebe covering theorem.*) *If* $f: S(0,1) \to U$ *is a nor-*

*malized equivalence, then* $\{w: |w| < 4^{-1}\} \subset f(S(0, 1))$. $\square$

LEMMA 3.9. *Let* $f: S(0, 1) \to U$ *be an equivalence of* $S(0, 1)$ *onto an open set* $U$ *and* $\varepsilon > 0$. *Suppose for some* $r \in (0, 1)$, $d(f(0), f(\Gamma(r))) \geqq \varepsilon$ *with* $\Gamma(r) \equiv \{z: |z| \equiv r\}$. *If* $\gamma$ *is path with left endpoint* $f(0)$ *and* $d^*(\gamma, f(\Gamma(r))) \geqq \varepsilon$, *then* $\gamma \subset\subset f(S(0, 1))$. $\square$

The next lemma gives us an internal characterization of proximity of points to the boundary of a set equivalent to $S(0, 1)$.

LEMMA 3.10. *Let* $f: S(0, 1) \to U$ *be a normalized equivalence. Then for each* $\varepsilon > 0$, *there exists* $R \in (0, 1)$ *such that for each* $z_0$ *with* $|z_0| = R$ *and each* $r \in (R, 1)$,

$$d(f(z_0), \{f(z): |z| = r\}) \leqq \varepsilon .$$

*Proof.* Let $\varepsilon \in (0, 1)$ be fixed. We use the notation

$$\Gamma(r) \equiv \{z: |z| = r\}$$

for each $r \in (0, 1)$. Since $f$ is normalized, it is easy to show that

$$(1) \qquad \rho(0, f(\Gamma(r))) \equiv \inf\{|f(z)|: z \in \Gamma(r)\} \leqq 1$$

for each $r \in (0, 1)$. According to the constructive theory of metric spaces, there exists a number

$$(2) \qquad \beta \in (4\varepsilon^{-1}, 8\varepsilon^{-1})$$

such that the sets

$$(3) \qquad A_j \equiv \{z \in \Gamma(r_j): |f(z)| \leqq \beta\} \qquad (j \geqq 1)$$

are compact for a sequence $\{r_j: j \geqq 1\} \subset (0, 1)$ with

$$(4) \qquad r_{j+1} > 2^{-1}(1 + r_j) .$$

Since $A_j$ is compact, the supremum

$$(5) \qquad M_j \equiv \sup\{|f'(z)|: z \in A_j\}$$

exists for each $j \geqq 1$. Now there exists a positive integer $N$ (the size of $N$ depends on (1)) such that if there exist points $\{z_j \in \Gamma(r_j): 1 \leqq j \leqq N\}$ with disks $\{S(f(z_j), 2^{-9}\varepsilon): 1 \leqq j \leqq N\}$ that are mutually disjoint, then $|f(z_j)| > 8\varepsilon^{-1}$ for some $j \in \{1, \cdots, N\}$. (The existence of $N$ is a consequence of the fact that the disk $S(0, 8\varepsilon^{-1})$ cannot contain an infinite number of mutually disjoint disks of the same radius $2^{-9}\varepsilon$.)

Suppose

$$M_j > 2^{-5}\varepsilon(1 - r_j)^{-1} \qquad (1 \leqq j \leqq N) . \tag{6}$$

Then for each $j \in \{1, \cdots, N\}$, (5) implies

$$|f'(z_j)| > 2^{-5}\varepsilon(1 - r_j)^{-1} \tag{7}$$

for some $z_j \in A_j$. Then observe that the function $h$ defined by

$$h(w) \equiv \frac{f(z_j + (r_{j+1} - r_j)w) - f(z_j)}{f'(z_j)(r_{j+1} - r_j)} \tag{8}$$

is an equivalence on $S(0, 1)$ because $f$ is an equivalence on

$$S(z_j, r_{j+1} - r_j) .$$

Moreover $h(0) = 0$ and $h'(0) = 1$. By Lemma 3.8,

$$\{w: |w| < 4^{-1}\} \subset h(S(0, 1)) . \tag{9}$$

By the definition (8) of $h$, we get from (9)

$$\left\{\zeta: \left|\frac{\zeta - f(z_j)}{f'(z_j)(r_{j+1} - r_j)}\right| < 4^{-1}\right\} \subset f(S(z_j, r_{j+1} - r_j)) .$$

Therefore

$$\begin{aligned} &\{\zeta: |\zeta - \zeta_0| < 4^{-1}|f'(z_j)|(r_{j+1} - r_j)\} \\ &\qquad \subset f(S(z_j, r_{j+1} - r_j)) \subset f(S(0, r_{j+1})) . \end{aligned} \tag{10}$$

In view of condition (4), we obtain from (10),

$$\{\zeta: |\zeta - f(z_j)| < 4^{-1}|f'(z_j)|2^{-1}(1 - r_j)\} \subset f(S(0, r_{j+1})) . \tag{11}$$

By the maximum principle, (11) implies

$$\rho(f(z_j), f(\Gamma(r_{j+1}))) \geqq 4^{-1}|f'(z_j)|2^{-1}(1 - r_j) . \tag{12}$$

In view of condition (7), (12) implies

$$\rho(f(z_j), f(\Gamma(r_{j+1}))) \geqq 2^{-8}\varepsilon . \tag{13}$$

Thus $S(f(z_j), 2^{-9}\varepsilon) \subset f(S(0, 1))$ for each $j \in \{1, \cdots, N\}$. Because $f$ is an equivalence, the sets $\{S(f(z_j), 2^{-9}\varepsilon): 1 \leqq j \leqq N\}$ are multually disjoint, in view of condition (13). By the definition of the integer $N$, there exists some $k \in \{1, \cdots, N\}$ such that

$$|f(z_k)| > 8\varepsilon^{-1} . \tag{14}$$

But $z_k$ was so chosen that $z_k \in A_k$. According to (2) and (3), we have

$$|f(z_k)| < \beta < 8\varepsilon^{-1} . \tag{15}$$

This contradiction of (15) against (14) implies that condition (6) is impossible. Hence

$$M_J < 2^{-4}\varepsilon(1 - r_J)^{-1} \tag{16}$$

for some $J \in \{1, \cdots, N\}$. With this integer $J$, let $R \equiv r_J$ and $z_0$ be any point on $\Gamma(R)$. We intend to show that $d(f(z_0), f(\Gamma(r))) \leqq \varepsilon$ for all $r \in (R, 1)$. Now either

$$|f(z_0)| > 4\varepsilon^{-1} \tag{17}$$

or

$$|f(z_0)| < \beta\,. \tag{18}$$

If (17) holds, then

$$2(1 + |f(z_0)|^2)^{-1/2} \leqq 2|f(z_0)|^{-1} < 2^{-1}\varepsilon\,. \tag{19}$$

Since $f$ is a normalized equivalence ($f(0) = 0$), $||f||_{\Gamma(r)} > |f(z_0)|$. Hence we can choose a point $z \in \Gamma(r)$ such that

$$|f(z)| > |f(z_0)| > 4\varepsilon^{-1}\,. \tag{20}$$

Inequality (20) allows us to compute

$$\begin{aligned} d(f(z_0), f(z)) &= 2|f(z_0) - f(z)|(1 + |f(z_0)|^2)^{-1/2}(1 + |f(z)|^2)^{-1/2} \\ &\leqq 2|f(z)|(1 + |f(z)|^2)^{-1/2}2(1 + |f(z_0)|^2)^{-1/2} \\ &\leqq 2 \cdot 2^{-1}\varepsilon = \varepsilon\,. \end{aligned} \tag{21}$$

The last inequality in (21) is obtained from (19). Clearly (21) implies

$$\begin{aligned} d(f(z_0), f(\Gamma(r))) &= \inf\{d(f(z_0), f(z)) \colon z \in \Gamma(r)\} \\ &\leqq \varepsilon\,. \end{aligned}$$

If (18) holds, then $z_0 \in A_J$. Therefore, according to (16),

$$|f'(z_0)| < 2^{-4}\varepsilon(1 - R)^{-1}\,. \tag{22}$$

Now suppose

$$\rho(f(z_0), f(\Gamma(r))) > 2^{-2}\varepsilon\,. \tag{23}$$

Then

$$S(f(z_0), 2^{-2}\varepsilon) \subset U$$

because of Lemma 3.9. Let $g\colon U \to S(0, 1)$ be the inverse to $f$ and let $\zeta_0 \equiv f(z_0)$. Then the function $\phi$ defined by

$$\phi(w) \equiv \frac{g(2^{-2}\varepsilon w + \zeta_0) - g(\zeta_0)}{g'(\zeta_0)2^{-2}\varepsilon} \tag{24}$$

is an equivalence on $S(0, 1)$ because $g$ is an equivalence on $S(\zeta_0, 2^{-2}\varepsilon)$. A simple computation shows that $\phi$ is normalized: $\phi(0) = 0$ and $\phi'(0) = 1$. By Lemma 3.8,

$$\{w: |w| < 4^{-1}\} \subset \phi(S(0, 1)) . \tag{25}$$

Interpreting (25) with respect to (24), we get

$$\{z: |z - z_0| < 2^{-4}\varepsilon |g'(\zeta_0)|\} \subset g(S(\zeta_0, 2^{-2}\varepsilon)) \subset g(U) = S(0, 1) . \tag{26}$$

Since $g'(\zeta_0)f'(z_0) = 1$, (26) implies

$$\{z: |z - z_0| < 2^{-4}\varepsilon |f'(z_0)|^{-1}\} \subset S(0, 1) . \tag{27}$$

Since $|z_0| = R$, (27) stipulates that $1 - R > 2^{-4}\varepsilon |f'(z_0)|^{-1}$ or

$$|f'(z_0)| > 2^{-4}\varepsilon(1 - R)^{-1} . \tag{28}$$

But (28) contradicts (22). Hence (23) is impossible. This means that

$$\rho(f(z_0)), f(\Gamma(r))) \leqq 2^{-2}\varepsilon . \tag{29}$$

Since $d(z, z') \leqq 2\rho(z, z')$ for all $z, z' \in \mathscr{C}$, we have from (29) the conclusion $d(f(z_0), f(\Gamma(r))) \leqq \varepsilon$. □

THEOREM 3.11. *If $f: S(0, 1) \to U$ is an equivalence of $S(0, 1)$ onto an open set $U$, then $U$ is mappable.*

*Proof.* Without loss in generality we can assume that $f$ is normalized, that is, $f(0) = 0$ and $f'(0) = 1$. $U$ is simply connected because $S(0, 1)$ is simply connected. It remains to show that $U$ has an $\varepsilon$-border for each $\varepsilon > 0$. To this end, choose 0 to be the distinguished point of $U$. For a fixed $\varepsilon > 0$ that is sufficiently small, there exists a sequence $\{r_j: j \geqq 1\}$ in $(0, 1)$ such that

$$0 < r_j < r_{j+1} < 1 \qquad (j \geqq 1) , \tag{1}$$

$$r_j \longrightarrow 1 \quad \text{as} \quad j \longrightarrow \infty , \tag{2}$$

$$d(f(z), f(\Gamma(r))) \leqq 2^{-(j+4)}\varepsilon \qquad (j \geqq 1) \tag{3}$$

for each $z \in \Gamma(r_j)$ and each $r \in (r_j, 1)$, and

$$d(0, f(\Gamma(r_0))) > 2^{-2}\varepsilon . \tag{4}$$

The construction of $\{r_j\}$ is carried out by repeated application of Lemma 3.10. Let $w_1, \cdots, w_n$ be a $2^{-4}\varepsilon$ approximation to $f(\Gamma(r_0))$ relative to the metric $d$, for some integer $n \geqq 1$. For each $k \in \{1, \cdots, n\}$, let $z_k^0 \equiv g(w_k)$, where $g$ is the inverse to $f$. By condition (3), we see that there exists a sequence $\{z_k^j: 0 \leqq j < \infty\}$ in $S(0, 1)$ such that

$$(5) \qquad z_k^j \in \Gamma(r_j)$$

and $d(f(z_k^j), f(z_k^{j+1})) \leqq 2^{-(j+4)}\varepsilon$ for each $k \in \{1, \cdots, n\}$. Hence each sequence $\{f(z_k^j): 0 \leqq j < \infty\}$ is Cauchy in $\mathscr{C}$ relative to $d$. Since $f$ is a normalized equivalence, $\rho(0, f(\Gamma(r))) \leqq 1$ for all $r \in (0, 1)$, and we see that at least one sequence $\{f(z_k^j): 0 \leqq j < \infty\}$ has a limit, say $\zeta_k \in \mathscr{C}$. From this fact, we can then assume that for each $k \in \{1, \cdots, n\}$, there exists $\zeta_k \in \mathscr{C}$ such that

$$(6) \qquad d(f(z_k^0), \zeta_k) \leqq 2^{-2}\varepsilon$$

and each $\zeta_k$ is the limit of at least one of the sequences $\{z_k^j: 0 \leqq j < \infty\}$. (Note that $\mathscr{C}$ relative to $d$ is not a complete metric space and so we could not conclude directly that the Cauchy sequences

$$\{\{z_k^j: 0 \leqq j < \infty\}: 1 \leqq k \leqq n\}$$

converge.)

We claim that $\zeta_k \in \sim U$ for each $k \in \{1, \cdots, n\}$. Suppose $\zeta \in U$. Then we have to show that $\zeta \neq \zeta_k$. Since $U$ is open, let $D(\zeta, r) \subset U$ for some $r > 0$. Suppose $|\zeta - \zeta_k| < r$. Then there exists $z \in S(0, 1)$ with $f(z) = \zeta_k$. Since $f(z_m^j) \to \zeta_k$ as $j \to \infty$, for some $m \in \{1, \cdots, n\}$, we conclude that $z_m^j \to z$ as $j \to \infty$. But conditions (5) and (2) together imply $|z_m^j| \to 1$ as $j \to \infty$. Therefore $|z| = 1$. This contradiction forces $|\zeta - \zeta_k| \geqq r$. Since $\zeta$ is an arbitrary point in $U$, we see that $\zeta_k \in \sim U$ for each $k \in \{1, \cdots, n\}$.

Finally, we claim that $\{\zeta_1, \cdots, \zeta_n\}$ form an $\varepsilon$-border for $U$ with distinguished point 0. Now let $\gamma$ be a path with left endpoint 0 and

$$(7) \qquad d^*(\gamma, \{\zeta_1, \cdots, \zeta_n\}) \geqq \varepsilon .$$

We have the triangle inequality:

$$\begin{aligned} &d^*(\gamma, w_k) + d(w_k, \zeta_k) \\ &\quad = \inf\{d(z, w_k): z \in \gamma\} + d(w_k, \zeta_k) \\ &\quad = \inf\{d(z, w_k) + d(w_k, \zeta_k): z \in \gamma\} \\ &\quad \geqq \inf\{d(z, \zeta_k): z \in \gamma\} \\ &\quad = d^*(\gamma, \zeta_k) . \end{aligned}$$

Therefore

$$\begin{aligned} d^*(\gamma, w_k) &\geqq d^*(\gamma, \zeta_k) - d(w_k, \zeta_k) \\ &\geqq \varepsilon - 2^{-2}\varepsilon > 2^{-1}\varepsilon , \end{aligned}$$

in view of (7) and (6). Since $\{w_1, \cdots, w_n\}$ is a $2^{-4}\varepsilon$ approximation to $f(\Gamma(r_0))$, we conclude that $d^*(\gamma, f(\Gamma(r_0))) \geqq 2^{-2}\varepsilon$. We use condition (4) to show that $\gamma \subset\subset U$, by Lemma 3.9. Hence $U$ is mappable.

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