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ON THE FRACTIONAL PARTS OF A SET OF POINTS II

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Heilbronn proved that for any $\varepsilon > 0$ there exists a number $C(\varepsilon)$ such that for any real numbers θ and $N \geqslant 1$ there is an integer n such that

$$1 \leqslant n \leqslant N \quad \text{and} \quad \|n^2\theta\| < C(\varepsilon) N^{-1/2+\varepsilon}$$

where $\|\alpha\|$ denotes the difference between α and the nearest integer, taken positively. The method depends on Weyl's estimates for trigonometric sums. The result was generalized by Davenport who obtained analogous results for polynomials which have no constant term.

The object here is to obtain a result for simultaneous approximations to quadratic polynomials $f_1, \cdots f_R$ having no constant term:

For any $\varepsilon > 0$ there is a number $C = C(\varepsilon, R)$ such that for any $N \geqslant 1$ there is an integer n such that

$$1 \leqslant n \leqslant N \quad \text{and} \quad \|f_i(n)\| < CN^{-1/g(R)+\varepsilon} \quad \text{for } i = 1, \cdots, R\,,$$

where $g(1) = 3$ and $g(R) = 4g(R-1) + 4R + 2$ for $R \geqslant 2$.

1. Introduction. In 1948 Heilbronn [4] proved the result stated above on the distribution of the sequence $n^2\theta$ (mod 1). This was generalized to polynomials which have no constant term by Davenport [2].

THEOREM. *Let $\varepsilon > 0$ and let R be a positive integer. Then there is a number $C = C(\varepsilon, R)$ such that for any quadratic polynomials $f_1, \cdots, f_R$ having no constant term, and for any $N \geqslant 1$, there is an integer n such that*

$$(1) \qquad 1 \leqslant n \leqslant N \quad \text{and} \quad \|f_i(n)\| < CN^{-1/g(R)+\varepsilon} \quad \text{for } i = 1, \cdots, R\,,$$

where

$$(2) \qquad g(1) = 3 \quad \text{and} \quad g(R) = 4g(R-1) + 4R + 2 \quad \text{for} \quad R \geqq 2,$$

the result being uniform in $f_1, \cdots, f_R$.

It can be readily verified by induction that an explicit formula for $g(R)$ is

$$(3) \qquad 18g(R) = 29 \cdot 4^R - 24R - 20\,, \qquad \text{for } R \geqq 2\,.$$

2. Preliminaries to the proof. The case $R = 1$ was proved by Davenport [2]. The theorem will be proved by induction on R, so we suppose the theorem is true for $R-1$. ε denotes a small positive number and $r(\varepsilon)$ denotes a multiple of ε depending only on R, note that $r(\varepsilon)$ differs in its various occurrences. We may suppose that $N > N_0(\varepsilon, R)$. $F \ll G$ means that $|F| < CG$ where C depends at most on ε and R. $e(z) = \exp(2\pi i z)$.

LEMMA 1 (*Vinogradov*). *Let Δ satisfy $0 < \Delta < 1/2$ and let a be a positive integer. Then there exists a function $\psi(z)$, periodic with period 1, which satisfies*

$$\psi(z) = 0 \qquad \text{for } \|z\| > \Delta \tag{4}$$

and

$$\psi(z) = \sum_{m=-\infty}^{\infty} a_m e(mz)$$

where the a_m are real numbers, $a_0 = \Delta$, $a_m = a_{-m}$, $m = 1, 2, \cdots$, and

$$|a_m| < A \min(\Delta, m^{-a-1}\Delta^{-a}) , \qquad m \neq 0 , \tag{5}$$

where A depends only on a.

Proof. This is a particular case of Lemma 12 of Chapter 1 of Vinogradov [5].

LEMMA 2 (Weyl). *Let A and P be real numbers, $P \geqslant 1$. Let $\alpha = aq^{-1} + \beta$ where $(a, q) = 1$, $q \geqslant 1$ and $|\beta| \leqq q^{-2}$. Then*

$$\left| \sum_{A \leqq n \leqq A+P} e(\alpha n^2 + \alpha_1 n) \right|^2 \ll P^{\varepsilon}(q^{-1}P + 1)(P + q \log q) . \tag{6}$$

Proof. See, for example, Lemma 1 of Davenport [1].

Let

$$f_i(n) = \theta_i n^2 + \phi_i n , \qquad i = 1, \cdots, R . \tag{7}$$

We choose a positive number δ so that there is no integer n with

$$1 \leqslant n \leqslant N \quad \text{and} \quad \|f_i(n)\| \leqslant N^{-\delta} , \qquad i = 1, \cdots, R . \tag{8}$$

We may suppose that $\delta < 1/g(R)$. We take $\Delta = N^{-\delta}$ and $a = [2\varepsilon^{-1}] + 1$ in Lemma 1. Then

$$\sum_{n=1}^{N} \prod_{i=1}^{R} \psi(f_i(n)) = 0$$

so

$$N^{1-R\delta} + \boldsymbol{\Sigma}^* a_{m_1} \cdots a_{m_R} T(\boldsymbol{m}) = 0$$

where Σ^* denotes a summation over $-\infty < m_1 < \infty, \cdots, -\infty < m_R < \infty$, $\boldsymbol{m} = (m_1, \cdots, m_R) \neq 0$,

$$T(\boldsymbol{m}) = \sum_{n=1}^{N} e(\boldsymbol{m}.\boldsymbol{\theta} n^2 + \boldsymbol{m}.\boldsymbol{\phi} n) , \tag{9}$$

$$\boldsymbol{m}.\boldsymbol{\theta} = \sum_{i=1}^{R} m_i\theta_i \quad \text{and} \quad \boldsymbol{m}.\boldsymbol{\phi} = \sum_{i=1}^{R} m_i\phi_i . \tag{10}$$

Summing over terms in the region $|m_1| > N^{\delta+\varepsilon}$ we have

$$\begin{aligned} \sum |a_{m_1} \cdots a_{m_R} T(\boldsymbol{m})| &\ll N \sum N^{a\delta} m_1^{-a-1} \\ &\ll N^{1-a\varepsilon} \end{aligned}$$

by Lemma 1, and similarly for other regions $|m_i| > N^{\delta+\varepsilon}$. Thus

$$\begin{aligned} 1 &\ll N^{-1+R\delta} \Sigma' |a_{m_1} \cdots a_{m_R} T(\boldsymbol{m})| \\ &\ll N^{-1} \Sigma' |T(\boldsymbol{m})| \end{aligned} \tag{11}$$

where Σ' denotes a summation over $\max|m_i| \leqslant N^{\delta+\varepsilon}$, $\boldsymbol{m} \neq \boldsymbol{0}$. Taking the square of this inequality and applying Cauchy's inequality we have

$$1 \ll N^{-2+R\delta+R\varepsilon} S \tag{12}$$

where

$$S = \Sigma' |T(\boldsymbol{m})|^2 . \tag{13}$$

We now proceed to estimate S. Let $Q = N^A$, $T = N^B$ where A and B will be chosen later. By Dirichlet's theorem on Diophantine approximation, see Theorem 185 of Hardy and Wright [3], for each $\boldsymbol{m}$ there exist integers a, b, q and t such that

$$\boldsymbol{m}.\boldsymbol{\theta} = aq^{-1} + \alpha \text{ with } (a, q) = 1 , \quad 1 \leqslant q \leqslant Q , \quad q|\alpha| \leqslant Q^{-1} \tag{14}$$

$$\boldsymbol{m}.\boldsymbol{\phi} = bt^{-1} + \beta \text{ with } (b, t) = 1 , \quad 1 \leqslant t \leqslant T , \quad t|\beta| \leqslant T^{-1} . \tag{15}$$

3. **The induction step.** For any $\boldsymbol{m}$ in the sum for S we have

$$\max|m_i| \leqslant N^{\delta+\varepsilon} . \tag{16}$$

Since $\boldsymbol{m} \neq \boldsymbol{0}$ and $|T(-\boldsymbol{m})| = |T(\boldsymbol{m})|$ we may suppose that $m_R > 0$. We take

$$\sigma = 2g(R-1)\delta + 4g(R-1)\varepsilon , \tag{17}$$

$$A = \frac{3}{2} + (2g(R-1)+1)\,\delta + (4g(R-1)+3)\varepsilon , \tag{18}$$

and

$$B = \frac{1}{2} + 2\varepsilon \,. \tag{19}$$

Applying the case $R-1$ of the theorem to the polynomials

$$f_i^*(n) = m_R q^2 t^2 \theta_i n^2 + qt\phi_i n,\ i = 1, \cdots, R-1 \,, \tag{20}$$

we see that there is an integer x such that

$$1 \leqslant x \leqslant N^\sigma \quad \text{and} \quad \|f_i^*(x)\| \ll N^{-\sigma/g(R-1)+\varepsilon} \,, \quad i = 1, \cdots, R-1 \,. \tag{21}$$

Suppose that $q < N^{1/2-\sigma-\delta-4\varepsilon}$. Taking $y = m_R qtx$ we have $1 \leqq y \leqq N$ and for $i = 1, \cdots, R-1$

$$\begin{aligned} \| f_i(y) \| &= \| m_R^2 q^2 t^2 \theta_i x^2 + m_R qt\phi_i x \| \\ &\leqslant | m_R | \, \|f_i^*(x)\| \leqslant N^{-\delta} \,, \end{aligned} \tag{22}$$

by (16), (17), and (21). Also

$$\begin{aligned} \|f_R(y)\| &= \| m_R^2 q^2 t^2 \theta_R x^2 + m_R qt\phi_R x \| \\ &\leqslant \| m_R q^2 t^2 x^2 \boldsymbol{m}.\boldsymbol{\theta} \| + \| \textstyle\sum m_i (m_R q^2 t^2 \theta_i x^2 + qt\phi_i x) \| \\ &\quad + \| \textstyle\sum m_i qt\phi_i x + m_R qt\phi_R x \| \\ &\leqslant | m_R q t^2 x^2 | \, \| q\boldsymbol{m}.\boldsymbol{\theta} \| + \textstyle\sum | m_i | \, \|f_i^*(x)\| \\ &\quad + | xq | \, \| t\boldsymbol{m}.\boldsymbol{\phi} \| \\ &\leqslant N^{-\delta} \,, \end{aligned} \tag{23}$$

by (14) – (21), where the summations are over $i = 1, \cdots, R-1$.

This contradicts the assumption that there were no integer solutions of (8). Therefore $q \geqslant N^{1/2-\sigma-\delta-4\varepsilon}$.

4. Completion of the proof of the theorem. From (6) we have

$$| T(\boldsymbol{m}) |^2 \ll q^{-1} N^{2+\varepsilon} + qN^\varepsilon + N^{1+\varepsilon} \,. \tag{24}$$

For $N^{1/2-\sigma-\delta-4\varepsilon} \leqslant q \leqslant N$ we have

$$| T(\boldsymbol{m}) |^2 \ll q^{-1} N^{2+\varepsilon} \ll N^{1+1/2+(2g(R-1)+1)\delta+r(\varepsilon)} \,. \tag{25}$$

Summing over $O(N^{R(\delta+\varepsilon)})$ such $\boldsymbol{m}$ we have a contribution S_1 to S where

$$S_1 \ll N^{1+1/2+(2g(R-1)+R+1)\delta+r(\varepsilon)} \,. \tag{26}$$

For $N \leqslant q \leqslant M = N^A$ we have

$$| T(\boldsymbol{m}) |^2 \ll qN^\varepsilon \ll N^{1+1/2+(2g(R-1)+1)\delta+r(\varepsilon)} \,. \tag{27}$$

Summing over $O(N^{R(\delta+\varepsilon)})$ such $\boldsymbol{m}$ we have a contribution S_2 to S where

$$S_2 \ll N^{1+1/2+(2g(R-1)+R+1)\delta+r(\varepsilon)} . \tag{28}$$

Therefore, from (12), we have

$$1 \ll N^{-1/2+(2g(R-1)+2R+1)\delta+r(\varepsilon)} . \tag{29}$$

Hence

$$-\varepsilon < -\frac{1}{2} + (2g(R-1) + 2R + 1)\delta + r(\varepsilon)$$

so

$$\delta > 1/g(R) - r(\varepsilon) \tag{30}$$

and the theorem is proved.

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