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RESTRICTING ISOTOPIES OF SPHERES

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In this note we consider the problem of determining whether isotopic homeomorphisms of S^n that agree on a subset X of S^n are isotopic by an isotopy that is fixed on X. In particular, in the PL category, an affirmative answer is obtained for X a locally unknotted closed cell or an unknotted sphere.

If X and Y are polyhedra and h_0 and h_1 are homeomorphisms of X onto Y, then an *isotopy* between h_0 and h_1 is a homeomorphism $H: X \times I \rightarrow Y \times I (I = [0, 1])$ such that $H(x, t) = (h_t(x), t)$ for all $(x, t) \in X \times I$. Two embeddings f, g of X in Y are said to be *ambient isotopic* if there is an isotopy $H: Y \times I \rightarrow Y \times I$ such that $h_0 = \text{id.}$, and $h_1 f = g$. The isotopy H is *fixed on* $A \subset Y$ if $H(x, t) = (x, t)$ for all $(x, t) \in A \times I$. Let S^n denote the standard n-sphere, E^n Euclidean n-space, Δ^k a k-simplex in some combinatorial triangulation of S^n or E^n, and let "PL" denote "piecewise linear." If $k < n$ we regard S^n as the $(n-k)$-fold suspension of S^k, so there is a natural inclusion $S^k \subset S^n$. A PL embedding $i: S^k \rightarrow S^n$ is *unknotted* if $(S^n, i(S^k)) \underset{\approx}{PL} (S^n, S^k)$, which is always the case if $k \leqq n - 3$. Clearly an unknotted sphere Σ^k in S^n is PL locally flat; i.e., for each point $x \in \Sigma^k$, there is a neighborhood U of x in S^n such that

$$(U, U \cap \Sigma^k) \underset{\approx}{PL} (E^n, E^k) .$$

The main results of this paper are the following:

THEOREM 1. *Let $X = \Delta^k$ or $X = S^k$, and let $i: X \rightarrow S^n$ be a PL-embedding, unknotted if $X = S^k$, locally unknotted if $X = \Delta^k$. If f and g are PL-homeomorphisms of S^n that are ambient isotopic, and if $f \mid i(X) = g \mid i(X)$, then f and g are PL ambient isotopic fixing $i(X)$.*

THEOREM 2. *Let $\Sigma^k \subset S^n$ be unknotted, $n \geqq 5$, $k \neq 3$, and f and g be homeomorphisms S^n that are isotopic and agree on Σ. Then f and g are ambient isotopic fixing Σ.*

If $k \leqq n - 3$, then Theorem 1 is a special case of [2]. Note that in Theorem 2, we do not require f and g to be PL.

The key step in the proof of these theorems is

LEMMA 3. *Let X be a k-simplex in S^n or the standard k-sphere $S^k \subset S^n$. If f is an orientation preserving PL-homeomorphism of S^n*

that is the identity on X, then f is PL-isotopic to the identity keeping X fixed.

Proof. The proof is by induction on n, with the case $n = 0$ trivial. Assume the lemma is true for X a simplex or a sphere in S^{n-1}.

Case 1. $X = \Delta^k$.

Let D be a second derived neighborhood of X mod ∂X; if $k = 0$, D is a regular neighborhood of X; if $k = n$, $D = X$. Observe that $f(D)$ is also such a regular neighborhood. Thus there is an isotopy H of S^n, keeping X fixed, such that $H_0 = \text{id.}$, and $H_1 f(D) = D$ [1].

Now $H_1 f \mid \partial D$ is an orientation preserving *PL* homeomorphism; since $H_1 f \mid (\partial D \cap X = S^{k-1}) = \text{id.}$, and $\partial D \underset{\approx}{PL} S^{n-1}$, by induction, $H_1 f \mid \partial D$ is isotopic (in ∂D) to the id. fixing $\partial D \cap X$. Thus there is a *PL* isotopy G'_t of ∂D such that $G'_0 = \text{id.}$, $G'_1 H_1 f \mid \partial D = id.$, and $G'_t H_1 f \mid \partial D \cap X = \text{id.}$ for $0 \leqq t \leqq 1$. Suspend this isotopy to obtain an isotopy, G, of S^n that keeps X fixed; to do this, pick suspension points $x \in X$, $y \in S^n \backslash D$, and note that we may assume that X is then a subcone of D. Thus G has similar properties to G'; i.e., $G_t H_1 f \mid \partial D \cup X = \text{id.}$ for $0 \leqq t \leqq 1$, and $G_0 = \text{id.}$ The *PL*-homeomorphism $G_1 H_1 f$ of S^n is the *id.* on $\partial D \cup X$, so the Alexander technique yields an isotopy F of S^n such that $F_0 = G_1 H_1 f$, $F_1 = id.$, and F keeps $\partial D \cup X$ fixed. The isotopy

$$
\begin{array}{ll}
H_{4t}(f(x)) & 0 \leqq t \leqq \dfrac{1}{4},\ x \in S^n\ , \\[2ex]
G_{4t-1}(H_1 f(x)) & \dfrac{1}{4} \leqq t \leqq \dfrac{1}{2},\ x \in S^n \\[2ex]
F_{2t-1}(x) & \dfrac{1}{2} \leqq t \leqq 1,\ x \in S^n
\end{array}
$$

is the required result.

Case 2. $X = S^0$

Let $X = \{a, b\}$, and let N be a second derived neighborhood of a mod b in S^n. Let M be a second derived neighborhood of b mod $(N \cup f(N))$ in S^n. Then N and $f(N)$ are regular neighborhoods of a in $Q = \text{cl}\ (S^n - M)$ that meet ∂Q regularly. Thus there exists an ambient isotopy H of Q, keeping $\partial Q \cup a$ fixed, such that $H_1 f(N) = N$. Extend H to S^n by the identity on M.

$H_1 f \mid \partial N \colon \partial N \to \partial N$ is an orientation preserving *PL* homeomorphism $(\partial N \overset{PL}{\approx} S^{n-1})$, so we may use the Alexander technique to obtain an ambient isotopy G of S^n such that $G_1 H_1 f = \text{id.}$ and $G_t \mid X = \text{id.}$ As before, this yields the desired result.

Case 3. $X = S^k$, $k \geqq 1$.

Clearly we may assume $k < n$, and that if $S^n = \Sigma^{n-1}S^1$, then $S^k = \Sigma^{k-1}S^1$. Let $a, b \in S^1 \subset S^n$, and let $S_*^{n-1} = \Sigma^{n-1}\{a, b\}$. (In each of these suspensions, we are using the same suspension points in the same order.) Let B_+^n, B_-^n be the closed complementary domains of S_*^{n-1}. Let $B_+^k = S^k \cap B_+^n$; $B_-^k = S^k \cap B_-^n$.

Observe that B_+^n is a regular neighborhood of B_+^k mod B_-^k, as is $f(B_+^n)$. Thus by Theorem 3 of [1], there exists an isotopy H of S^n, fixed on $B_+^k \cup B_-^k = S^k$, such that

$$H_0 = \text{id}, \quad \text{and} \quad H_1 f(B_+^n) = B_+^n \ .$$

Note that $H_1 f(S_*^{n-1}) = S_*^{n-1}$, and that

$$H_1 f \mid S^k \cap S_*^{n-1} \colon S^k \cap S_*^{n-1} (= S^{k-1}) \longrightarrow S_*^{n-1} \text{ is the id} \ .$$

Thus $H_1 f \mid S_*^{n-1}$ is isotopic to the identity keeping $S^k \cap S_*^{n-1}$ fixed. Proceed as before to complete the proof.

COROLLARY 4. *If f is an orientation preserving PL homeomorphism of E^n such that $f \mid \Delta^k =$ id., then f is PL-isotopic to the* id. *fixing Δ^k.*

COROLLARY 5. *Let $g\colon \Delta^k \longrightarrow E^n(S^n)$ be a PL-embedding, locally unknotted if $k = n - 2$. If f is an orientation preserving PL-homeomorphism of $E^n(S^n)$ and if $f \mid g(\Delta) =$* id.*, then f is PL-isotopic to the identity fixing $g(\Delta)$.*

COROLLARY 6. *Let $g\colon S^k \longrightarrow S^n$ be an unknotted PL-embedding. If f is a PL-homeomorphism of S^n that is orientation preserving and the identity on $g(S^k)$, then f is PL-isotopic to the identity fixing $g(S^k)$.*

Proof of Theorem 1. Observe that gf^{-1} is an orientation preserving PL-homeomorphism of S^n that is the identity on $i(X)$. Thus there is a PL ambient isotopy h_t of S^n such that

$$\begin{aligned} h_0 &= \text{id} \ . \\ h_1 &= gf^{-1}, \text{ and} \\ h_t \mid i(X) &= \text{id} \ . \end{aligned}$$

$h_t\colon S^n \longrightarrow S^n$ is the desired isotopy.

Proof of Theorem 2. As in the proof of Theorem 1, it suffices to consider the case when f is orientation preserving and g is the identity. By [3], f is isotopic to a PL-homeomorphism f' fixing Σ^k. Apply Lemma 3 to f'.

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