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BOUNDEDLY HOLOMORPHIC CONVEX DOMAINS

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A boundedly holomorphic convex domain is a holomorphically convex domain with respect to the algebra of bounded holomorphic functions in the domain. The followings are shown in this paper: In a Riemann domain, a boundedly holomorphic convex domain is a domain of bounded holomorphy. With some restrictions, the converse is true. The spectrum of the algebra B of bounded holomorphic functions is an envelope of bounded holomorphy provided that the completion of B with the topology of uniform convergence on compact subsets is stable under differentiation. Finally, Stein manifolds of bounded type are introduced.

Let (X_1, A_1) and (X_2, A_2) be complex (analytic) manifolds. A map $\alpha: X_1 \to X_2$ said to be *biholomorphic* if α is a homeomorphism of X_1 onto X_2 and both α and α^{-1} are holomorphic. α is called a *spread map* if α is a locally biholomorphic. We denote a complex manifold $(X, A; \alpha)$ with a spread map α. A Riemann domain is a complex manifold which spreads in $(\boldsymbol{C}^n, \mathscr{O})$. We denote $B(X)$ for the algebra of all bounded holomorphic functions on X.

DEFINITION 1. Let (X, A) be a complex manifold and D be open in X. Let $B = B(D)$. D is said to be *boundedly holomorphic convex* if $\mathrm{hull}_B K = \hat{K}_B = \{x \in D; |f(x)| \leqq \|f\|_K \text{ for all } f \in B\}$ is compact provided K is a compact subset of D.

An open set D of X is called a *region of bounded holomorphy* if there is an $f \in B(D)$ for which every boundary point of D is a boundary singularity in the sense that f has no bounded analytic continuation to any open neighborhood of any boundary point (see [5]).

The following natural questions arise; if boundedly holomorphic convex domains are domains of bounded holomorphy, and vice versa. The answer for the first is affirmative.

LEMMA 1. *Let $\{\alpha_n\}$ be a sequence of complex numbers such that $|\alpha_n| \leqq 1$. And let $\{f_n\}$ be a sequence of bounded complex functions on a set X such that $\sum |f_n(x)|$ converges uniformly on x. Then the infinite product*

$$f(x) = \prod_{n=1}^{\infty} (\alpha_n + f_n(x))$$

converges uniformly on X.

THEOREM 1. *Let $(X, A; \alpha)$ be a Riemann domain. If it is boundedly holomorphic convex then it is a domain of bounded holomorphy.*

Proof. Assume that X is connected. Write $X = \bigcup_{n=1}^{\infty} \hat{K}_n$, where $K_n \subset K_{n+1}$ and $\hat{K}_n = \{x \in X; |f(x)| \leqq \|f\|_{K_n}$ for all $f \in B(X)\}$. Let $\{x_k\}$ be a countable dense set in X, and let $\Delta_k = \Delta(x_k, d(x_k))$, where $d(x) = \sup\{r; \Delta(x, r)\}$; $\Delta(x, r)$ is the neighborhood of x which is homeomorphic by α onto a polydisc with center $\alpha(x)$ and radius r in C^n. Let $\{S_n\}$ be a sequence

$$\Delta_1, \Delta_1, \Delta_2, \Delta_1, \Delta_2, \Delta_3, \Delta_1, \Delta_2, \cdots$$

such that each Δ_k occurs in $\{S_n\}$ infinitely many times. Take $y_n \in S_n - \hat{K}_n$ and choose a function $g_n \in B(X)$ such that $\|g_n\|_X = 1$ and $|g_n(y_n)| > \|g_n\|_{K_n}$. By taking sufficiently high power of g_n we can find f_n such that $\|f_n\|_X = 1$, $|f_n(y_n)| \leqq 1/9$, and $\|f_n\|_{K_n} < 1/(n \cdot 2^{l_n})$, where $l_n \geqq n$ and $l_n \in Z_+$. Also, by induction, we can arrange $l_{n+1} > l_n$. Put $f_n = (9/10) f_n$ (for simplicity use the same f_n), then $\|f_n\|_2 = 9/10$, $|f_n(y_n)| \leqq 1/10$, and $\|f_n\|_{K_n} < (9/10) \cdot (1/n \cdot 2^{l_n})$. Put $f_n(y_n) = \alpha_n$ and consider $\prod_{n=1}^{\infty} (\alpha_n - f_n)^{l_n}$. Since

$$\textstyle\sum_{n=1}^{\infty} n \cdot \|f_n\|_{K_n} < (9/10) \cdot 2^{-l_n} < \infty, \prod_{n=1}^{\infty} (\alpha_n - f_n)^{l_n}$$

converges uniformly on compact subsets of X to a holomorphic function f. And since $\|\alpha_n - f_n\|_X \leqq 1, f \in B(X)$. Furthermore, $\|f_n\|_{K_1} < \alpha_n$ for all n so that $(\alpha_n - f_n) | K_1$ is a unit of $C(K_1)$; the Banach algebra of continuous complex functions on K_1. Thus

$$f | K_1 = \lim_{\nu \to \infty} \prod^{\nu} (\alpha_n - f_n)^{l_n} | K_1 ;$$

the uniform limit in $C(K_1)$, is a unit of $C(K_1)$. Thus $f \neq 0$ in X.

Suppose for some $y \in X$, $\operatorname{rad}(f, y) > d(y) + \varepsilon$, where $\operatorname{rad}(f, y)$ is the radius of convergence of f at y. Since $\{x_k\}$ is a dense subset of X we can choose x_k close enough to y such that $\operatorname{rad}(f, x_k) > d(x_k) + \varepsilon/3$ and the power series $\hat{f}$ at x_k converges in $\Delta(\alpha(x_k), d(x_k) + \varepsilon/3)$. Since every Δ_k occurs in $\{S_n\}$ infinitely often $\Delta(x_k, d(x_k))$ contains infinitely many points y_n, and f has a zero of order n at y_n. Choose $\{s_n\} \subset \{\alpha(y_n)\} \subset \Delta(\alpha(x_k), d(x_k))$ such that $s_n \to s_0 \in \bar{\Delta}(\alpha(x_k), d(x_k))$. Then, by continuity, for any $k_1, \cdots, k_n$

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \hat{f}(s_0) = \lim_{n \to \infty} \frac{\partial^{k_1+\cdots+k_n}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \hat{f}(s_n) = 0 .$$

Hence the power series expansion $\hat{f}$ at s_0 is identically zero, so $\hat{f} \equiv 0$, which is a contradiction. Thus $\operatorname{rad}(f, y) = d(y)$ for all $y \in X$. Since $f \in B(X)$, X is a domain of bounded holomorphy.

For the second question we raised; if a domain of bounded holomorphy is boundedly holomorphic convex, we first give a couple of examples of domains of bounded holomorphy which are not boundedly holomorphic convex.

EXAMPLE 1.[1] Let $D = \{(z, \omega) \in \boldsymbol{C}^2; |\omega| < e^{-|z|^2}\}$. If $f_n \in \mathcal{O}(\boldsymbol{C})$, $|f_n| \leqq k_n e^{n|z|^2}$ with $\Sigma k_n < \infty$, then $\Sigma f_n(z)\omega^n$ is bounded in D. It follows that D is a domain of bounded holomorphy but it is not boundedly holomorphic convex.

EXAMPLE 2. Let $D = \{(z, \omega) \in \boldsymbol{C}^2; |z| < |\omega| < 1\}$. Then D is a domain of bounded holomorphy but it is not boundedly holomorphic convex.

Thus we need some restrictions to claim an affirmative answer for the question.

Let (X, A) be a complex manifold and $B = B(X)$. Denote $\tilde{B}$ for the closure of B in $C(X)$ with the topology of uniform convergence on compact subsets of X (abbreviate as c.o. topology). If all partial derivatives of functions in $\tilde{B}$ are in $\tilde{B}$, $\tilde{B}$ is said to be stable under differentiation.

Although the following propositions and their proofs are analogue to those for unbounded functions (see Katznelson [4] or Gunning and Rossi [3]), they have their own right for bounded functions.

PROPOSITION 1. *Let $(X, A; \alpha)$ be a Riemann domain such that $d(K) < \infty$ for a compact subset K of X, and let $B = B(X)$. Suppose $\tilde{B}$ is stable under differentiation. Then* (1) *implies* (2) *and, in addition, if X is finitely sheeted* (2) *implies* (3).

(1) *$(X, A; \alpha)$ is a domain of bounded holomorphy,*

(2) *$d(K) = d(\hat{K}_B)$ for all compact subsets K of X,*

(3) *$(X, A; \alpha)$ is boundedly holomorphic convex.*

NOTE. If D is merely a domain of holomorphy then (2) need not be true; let $D = \Delta(0, 1) - \{0\}$, the punctured open unit disc in $\boldsymbol{C}$, and K be the circle with center 0 and radius 1/3. Then $\hat{K}_A = K$ and $d(K) = d(\hat{K}_A) = 1/3$. On the other hand $\hat{K}_B = \bar{\Delta}(0, 1/3) - \{0\}$ and $d(\hat{K}_B) = 0$.

PROPOSITION 2. *Let $(X, A; \alpha)$ be a Riemann domain and α be a bounded spread map; i.e., $f_1, \cdots, f_n \in B(X)$ where $\alpha = (f_1, \cdots, f_n)$. Suppose that $B(X)$ separates the points of X. If $d(K) = d(\hat{K}_B)$ for all compact subsets K of X then X is boundedly holomorphic convex.*

[1] Example 1 is provided by the referee.

NOTE. Let D be a domain in Example 2 and $B = B(D)$. Then $\tilde{B}$ is an algebra which is not stable under differentiation, for, if so D is boundedly holomorphic convex by Propositions 1 and 2.

DEFINITION 2. Let $R(\boldsymbol{C}^n, \mathcal{O})$ be the category whose objects are complex manifolds $(X_\nu, A_\nu; \alpha_\nu)$ spread in $(\boldsymbol{C}^n, \mathcal{O})$ and morphisms are spread maps $\beta_\mu: (X_\nu, A_\nu; \alpha_\nu) \to (X_\mu, A_\mu; \alpha_\mu)$ such that $\alpha_\nu = \alpha_\mu \circ \beta_\mu$. Let $S(X, A; \alpha, F)$ be the class of $(\beta_\nu; X_\nu, A_\nu; \alpha_\nu)$ such that $\beta_\nu: (X, A; \alpha) \to (X_\nu, A_\nu; \alpha_\nu)$ is a morphism with $\alpha = \alpha_\nu \circ \beta_\nu$ and $\beta_\nu^* A_\nu(X_\nu) \supset F$, where β_ν^* is the adjoint of β_ν. We define a quasi-order in $S(X, A; \alpha, F)$ as follows: $(\beta_\nu; X_\nu, A_\nu; \alpha_\nu) < (\beta_\mu; X_\mu, A_\mu; \alpha_\mu)$ if and only if there exists a morphism $\gamma: (X_\nu, A_\nu; \alpha_\nu) \to (X_\mu, A_\mu; \alpha_\mu)$ such that $\beta_\mu = \gamma \circ \beta_\nu$ and $\alpha_\nu = \alpha_\mu \circ \gamma$. With respect to this quasi-order there is a maximal object which is unique within biholomorphic morphism. We call this object the F-envelope of holomorphy of $(X, A; \alpha)$. If $F = B(X)$ we call it the *envelope of bounded holomorphy.*

NOTE. The continuation of every function in $B(X)$ to the envelope is still bounded. To see this, apply the same argument on the envelope as in Lemma 15 of Kim [5].

A Bishop's theorem for an analytic structure on the spectrum of the algebra $A(X)$ carries in the following way over the bounded holomorphic functions. Although the proof is analogous to the case for unbounded functions we put it in detail. We are indebted for this proof to Quigley [7].

Let (X, A) be a complex manifold and $B = B(X)$ with the c.o. topology. The spectrum S_B of B is the set of all nontrivial continuous complex homomorphisms of B onto $\boldsymbol{C}$.

THEOREM 2. *Let $(X, A; \alpha)$ be a separable Riemann domain. Suppose $f_i \in B$, $i = 1, 2, \cdots, n$, where $\alpha = (f_1, \cdots, f_n)$, and $\tilde{B}$ is stable under differentiation. Then S_B is the envelope of bounded holomorphy.*

Proof. We observe that $S_B = S_{\tilde{B}}$ in set-wise and also topologically. Put $S = S_B$. Define $\hat{\alpha}: S \to \boldsymbol{C}^n$ by $\hat{\alpha}(\varphi) = (\varphi(f_1), \cdots, \varphi(f_n))$ for $\varphi \in S$, and $\rho: X \to S$ by $\rho(x) = \pi_x$, where $\pi_x(f) = f(x)$ for all $f \in B$. We will show that there is an analytic structure $\tilde{A}$ on S so that $(\rho; S, \tilde{A}; \hat{\alpha}, \hat{B})$ is the envelope of bounded holomorphy for $(X, A: \alpha, B)$.

(i) We claim that for $\varphi \in S$ there is a compact set $K \subset X$ such that $|\varphi(g)| \leqq \|g\|_K$ for all $g \in B$. If it is not true, let $X = \bigcup_{n=1}^\infty K_n$, where K_n are compact and $K_n \subset K_{n+1}$, then we have $g_n \in B$ such that $|\varphi(g_n)| = 1$ and $\|g_n\|_{K_n} < 2^{-n}$. Then Σg_n converges to an element of $\tilde{B}$, but since φ is continuous on $\tilde{B}$, $\varphi(\Sigma g_n) = \Sigma \varphi(g_n)$. The latter series is not convergent, which is absurd. Thus there is a compact set K such that $|\varphi(g)| \leqq ||g||_K$. Denote this K, $K = K_\varphi$.

(ii) Let $\boldsymbol{P}_\varphi = \{s \in \boldsymbol{C}^n; ||s - \hat{\alpha}(\varphi)||_\infty < d(K_\varphi)\}$ and with ε, $0 < \varepsilon < d(K_\varphi)$ let $\boldsymbol{P}_{\varphi,\varepsilon} = \{s \in \boldsymbol{C}^n\colon ||s - \hat{\alpha}(\varphi)||_\infty < \varepsilon\}$. For $f \in B$, $\varphi \in S$, and $s \in \boldsymbol{P}_\varphi$ we define the functional

$$L(f, \varphi, s) = \Sigma j!^{-1}\varphi\Big(\partial\frac{j}{\alpha}f\Big)(s_1 - \hat{\alpha}(\varphi)_1)^{j_1} \cdots (s_n - \hat{\alpha}(\varphi)_n)^{j_n} ,$$

where

$$\partial\frac{j}{\alpha}f = \frac{\partial_\alpha^{j_1+\cdots+j_n}}{\partial z_1^{j_1} \cdots \partial z_n^{j_n}}(f) \quad \text{and} \quad j! = j_1 \cdot j_2 \cdots j_n .$$

Then by a proposition in several complex variables and the continuity of φ,

$$\Big|\varphi\Big(\partial\frac{j}{\alpha}f\Big)\Big| \leqq \Big|\Big|\partial\frac{j}{\alpha}f\Big|\Big|_{K_\varphi} \leqq \frac{j!}{\varepsilon^{j_1+\cdots+j_n}}||f||_{(K_\varphi)_\varepsilon}$$

for all $f \in B(X)$, where $(K_\varphi)_\varepsilon = \boldsymbol{U}\{\bar{P}(x, \varepsilon); x \in K_\varphi\}$. Hence $L(f, \varphi, z)$ converges uniformly in z on $\bar{\boldsymbol{P}}_{\varphi,\varepsilon}$ for all $f \in B$ so that $L(\cdot, \varphi, z)$ is analytic in $\boldsymbol{P}_\varphi$. In particular, $L(\cdot, \varphi, s)$ is continuous for all $\varphi \in S$ and $s \in \boldsymbol{P}_\varphi$. Furthermore, since φ and $\partial(j/\alpha)$ are linear, $L(\cdot, \varphi, s)\colon B \to \boldsymbol{C}$ is linear and by using Leibnitz' formula:

$$k!^{-1}\partial\frac{k}{\alpha}(f \cdot g) = \sum_{i_1+j_1=K_1} \cdots \sum_{i_n+j_n=K_n} i!^{-1}\Big(\partial\frac{i}{\alpha}f\Big)j!\Big(\partial\frac{j}{\alpha}g\Big) ,$$

we know that $L(\cdot, \varphi, s)$ is multiplicative. Therefore, $L(\cdot, \varphi, s) \in S$ for $\varphi \in S$ and $s \in \boldsymbol{P}_\varphi$. Let $\varphi_s = L(\cdot, \varphi, s)$, then $L(\cdot, \varphi, s)$ has the following properties;

(a) $\varphi_{\hat{\alpha}(\varphi)} = L(\cdot, \varphi, \hat{\alpha}(\varphi)) = \varphi$

(b) $\varphi_s(f_j) = L(f_j, \varphi, s) = s_j$, $1 \leqq j \leqq n$ for $\alpha = (f_1, \cdots, f_n)$

(c) $\hat{\alpha}(\varphi_s) = s$.

(iii) We topologize S as follows: Let$Q_\varphi = \{\varphi_s; s \in \boldsymbol{P}_\varphi\}$, and $Q_{\varphi,\varepsilon} = \{\varphi_s\colon s \in \boldsymbol{P}_{\varphi,\varepsilon}\}$. Take the family $\{Q_{\varphi,\varepsilon}; \varphi \in S, 0 < \varepsilon\ d(K_\varphi)\}$ to be a subbase then this family gives a topology $\mathscr{T}$ on S. We show that $\mathscr{T}$-topology on S is equivalent to the weak topology on S induced by functions $\hat{f}$, where $\hat{f}$ is the Gelfand transform of f. First, $\mathscr{T}$-topology is finer than the weak topology: It suffices to show that all $\hat{f}$ in $\hat{B}$ are $\mathscr{T}$-continuous. For a given $\delta > 0$ we must find $\varepsilon > 0$ such that $\varphi_s \in Q_{\varphi,\varepsilon} \Rightarrow |\hat{f}(\varphi_s) - \hat{f}(\varphi)| < \delta$, or equivalently such that

$$s \in \boldsymbol{P}_{\varphi,\varepsilon} \Rightarrow |\varphi_s(f) - \varphi(f)| < \delta ,$$

or equivalently such that

$$s \in \boldsymbol{P}_{\varphi,\varepsilon} \Rightarrow |L(f, \varphi, s) - L(f, \varphi, \hat{\alpha}(\varphi)| < \delta .$$

Let $h(z) = L(f, \varphi, z) - L(f, \varphi, \hat{\alpha}(\varphi)) = L(f, \varphi, z) - \varphi(f)$; then h is analytic in $\boldsymbol{P}_\varphi$; in particular, h is continuous on $\boldsymbol{P}_\varphi$. Hence there exists $\varepsilon > 0$ such that $s \in \boldsymbol{P}_{\varphi,\varepsilon} \Rightarrow |h(s)| < \delta$ which shows that $\hat{f} \in \hat{B}$ are $\mathscr{T}$-continuous. Secondly, the weak topology is finer than $\mathscr{T}$-topology: It suffices to show that for $\varphi_s \in Q_{\varphi,\varepsilon}$ there is an open neighborhood of φ_s (w.r.t. the weak topology) which is contained in $Q_{\varphi,\varepsilon}$. We first observe that $\|\hat{\alpha}(h) - \hat{\alpha}(\varphi_s)\|_\infty = \|(h(f_1) - \varphi_s(f_1), \cdots, h(f_n) - \varphi_s(f_n))\|_\infty$. Let ε_1 be the radius of maximal polydisc with center $\hat{\alpha}(\varphi_s) = s$ which is contained in $\boldsymbol{P}_{\varphi,\varepsilon}$. If $\|\hat{\alpha}(h) - \hat{\alpha}(\varphi_s)\|_\infty < \varepsilon_1$ then $\hat{\alpha}(h) \in \boldsymbol{P}_{\varphi,\varepsilon}$ and so $h = h_{\hat{\alpha}(h)} \in Q_{\varphi,\varepsilon}$. Now $\|\hat{\alpha}(h) - \hat{\alpha}(\varphi_s)\|_\infty < \varepsilon_1$ iff $|h(f_i) - \varphi_s(f_i)| < \varepsilon_1$ for $1 \leqq i \leqq n$. Hence the open neighborhood $U_{\varphi_s} = \{h \in S: |\hat{f}_i(h) - \hat{f}_i(\varphi_s))| < \varepsilon_1, 1 \leqq i \leqq n\}$ is contained in $Q_{\varphi,\varepsilon}$, which shows $Q_{\varphi,\varepsilon}$ is open w.r.t. the weak topology.

(iv) Since $\hat{\alpha}(\varphi_s) = s$, $\hat{\alpha} | \varphi_{\varphi,\varepsilon}$ is one-to-one and $\hat{\alpha} | \varphi_{\varphi,\varepsilon} = \boldsymbol{P}_{\varphi,\varepsilon}$. And $\varphi = \varphi_{\hat{\alpha}(\varphi)} \in Q_{\varphi,\varepsilon}$. It follows that $\hat{\alpha}: S \rightarrow \boldsymbol{C}^n$ is a local homeomorphism. Let $\tilde{A} = \hat{\alpha}^* \mathscr{O}$ (the analytic structure on S induced by $\hat{\alpha}$), then $(S, \tilde{A}; \hat{\alpha})$ is a complex analytic manifold. Note that $\hat{B} = \{\hat{f}; f \in B\} \subset \tilde{A}$. For $\varphi \in S, s \in \boldsymbol{P}_\varphi$, we have $(\hat{\alpha} | Q_\varphi)^{-1}(s) = \varphi_s$ so that $\hat{f} \circ (\hat{\alpha} | Q)^{-1}(s) = \hat{f}(\varphi_s) = \varphi_s(f) = L(f, \varphi, s)$. Thus $\hat{f} \circ (\hat{\alpha} | Q_\varphi)^{-1}(z) = L(f, \varphi, z)$ which is convergent in $\boldsymbol{P}_\varphi$.

(v) $(S, \tilde{A}; \hat{\alpha}, \hat{B})$ is a B-continuation of $(X, A; \alpha, B)$. Let π_x be defined by $\hat{f}(\pi_x) = f(x)$ for $x \in X, f \in B$. Then $\pi_x \in S$ and $K_{\pi_x} = \{x\}$. Define $\rho: X \rightarrow S$ by $\rho(x) = \pi_x$. If T_x is the α-polydisc of center x and radius $d(K_{\pi_x}) = d(x)$ then $\alpha | T_x$ is one-to-one and $\alpha T_x = \boldsymbol{P}_{\pi_x}$. In fact $\hat{\alpha} \circ \rho(x) = \hat{\alpha}(\pi_x) = (\pi_x(f_1), \cdots, \pi_x(f_n)) = (f_1(x), \cdots, f_n(x) = \alpha(x)$ so that $\hat{\alpha} \circ \rho = \alpha$. We have shown that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\rho"] \arrow[dr, "\alpha"'] & & S \arrow[dl, "\hat{\alpha}"] \\ & \boldsymbol{C}^n & \end{tikzcd} .$$

Now we have to show $\hat{f} \circ \rho = f$ for every $f \in B$. We first observe that for each $q \in T_x$, $L(\cdot, \pi_x, \alpha(q)) = \pi_q$. For $\hat{\alpha}(\pi_x) = \alpha(x)$ and so

$$\begin{aligned} L(f, \pi_x, \alpha(q)) &= \Sigma j!^{-1} \pi_x\Big(\partial \frac{j}{\alpha} f\Big)(\alpha(q)_1 - \alpha(x)_1)^{j_1} \cdots (\alpha(q)_n - \alpha(x)_n)^{j_n} \\ &= \Sigma j!^{-1}\Big(\partial \frac{j}{\alpha} f\Big)(x)(\alpha(q)_1 - \alpha(x)_1)^{j_1} \cdots (\alpha(q)_n - \alpha(x)_n)^{j_n} \\ &= f(q) \\ &= \pi_q(f) . \end{aligned}$$

$Q_{\pi_x} = \{(\pi_x)_s; s \in \boldsymbol{P}_{\pi_x}\} = \{\pi_q; q \in T_x\}$, so that ρ maps T_x one-to-one and onto Q_{π_x}. Thus $\rho: X \rightarrow S$ is a local homeomorphism, and for $f \in B$, $\hat{f} \circ \rho(x) = \hat{f}(\pi_x) = \pi_x(f) = f(x)$ for all $x \in X$. Hence $\hat{f} \circ \rho = f$ for all

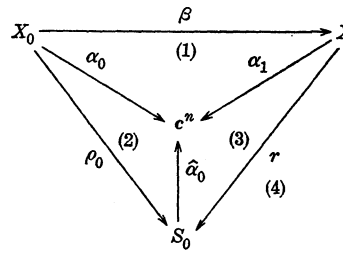

FIGURE 1

$f \in B$. Note that $\hat{f}$ is bounded on S for all $f \in B$.

(vi) Let X_0 be a connected component of X and S_0 be the connected component of S containing ρX_0. Let $f_0, \alpha_0, \rho_0, A_0, B_0$ be the restrictions of f, α, ρ, A, B to X_0 and $\hat{f}_0, \hat{\alpha}_0, \tilde{A}_0, \hat{B}_0$ be the restrictions of $\hat{f}, \hat{\alpha}, \tilde{A}, \hat{B}$ to S_0. Then $(S_0, \tilde{A}_0; \hat{\alpha}_0, \hat{B}_0)$ is a B_0-continuation of $(X_0, A_0; \alpha_0, B_0)$. Moreover, we claim that $(S_0, \tilde{A}_0; \hat{\alpha}_0, \hat{B}_0)$ is a maximal B-continuation. Take $(\beta, X_1, A_1; \alpha_1, B_1) \in S(X_0, A_0; \alpha_0, B_0)$ such that image $\beta^* \supset B_0$, i.e., for $f_0 \in B_0$ there is a unique $f_1 \in B_1$ such that $f_1 \circ \beta = f_0$. To show the above diagrams commute:

Define $r\colon X_1 \to S_0$ by $r(x_1)(f_0) = f_1(x_1)$, $x_1 \in X_1$. The commutativity of (1) is given and we have shown the commutativity of (2). To show (4), $r \circ \beta = \rho_0$; for $x_0 \in X_0$ and $f_0 \in B_0$, $r \circ \beta(x_0)(f_0) = f_1(\beta(x_0)) = f_0(x_0) = \pi_{x_0}(f_0) = \rho_0(x_0)(f_0)$. Hence $r \circ \beta = \rho_0$. To show (3), $\hat{\alpha}_0 \circ r = \alpha_1$. Let $\alpha_0 = (f_{10}, \cdots, f_{n0})$ and $\alpha_1 = (f_{11}, \cdots, f_{n1})$ then $\hat{\alpha}_0 \circ r(x_1) = (r(x_1)(f_{10}), \cdots, r(x_1)(f_{n0})) = (f_{11}(x_1), \cdots, f_{n1}(x_n))$ for $x_1 \in X_1$. It $x_1 \in \beta(X_0)$, i.e., $x_1 = \beta x_0$ for some $x_0 \in X_0$ then $\hat{\alpha}_0 \circ r(x_1) = \hat{\alpha}_0 r \beta(x_0) = (f_{11}\beta(x_0), \cdots, f_{n1}\beta(x_0)) = (f_{10}(x_0), \cdots, f_{n0}(x_0)) = \alpha_0(x) = \alpha_1\beta(x_0) = \alpha_1(x_1)$. Thus $\hat{\alpha}_0 \circ r = \alpha_1$ on $\beta X_0 \subset X_1$, and since X_1 is connected, $\hat{\alpha}_0 \circ r = \alpha_1$ on X_1.

To show that $(S_0, \tilde{A}_0; \hat{\alpha}_0, \hat{B}_0)$ is B-continuation of $(X_1, A_1; \alpha_1, \beta_1)$. For $x_1 \in X_1$, let $r(x_1) = \varphi \in S_0$. Let $Q_\varphi \subset S_0$. Take an α_1-polydisc T_1 of center x_1 such that $\alpha_1 T_1 = \boldsymbol{P}_{\varphi,\varepsilon}(\alpha_1(x_1) = \hat{\alpha}_0 \circ r(x_1) = \hat{\alpha}_0(\varphi)$ is the center). Then by (3), $rT_1 = Q_{\varphi,\varepsilon}$ and $r \mid T_1$ is one-to-one since $\alpha_1 \mid T_1$, $\hat{\alpha} \mid Q_{\varphi,\varepsilon}$ are one-to-one. Thus r is a local homeomorphism. Now we know that $f_1 \circ \beta = f_0 = \hat{f}_0 \circ \rho_0$, thus $\hat{f}_0 \circ r \circ \beta = \hat{f}_0 \circ \rho_0 = f_0 = f_1 \circ \beta$. Hence $\hat{f}_0 \circ r = f_1$ on $\beta X_0 \subset X_1$. Since X_1 is connected $\hat{f}_0 \circ r = f_1$ on X_1. We have shown that the following diagrams are commutative:

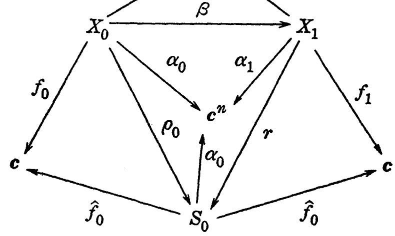

FIGURE 2

It follows that $(S_0, \tilde{A}_0; \hat{\alpha}_0, \hat{B}_0)$ is the B-envelope of holomorphy of $(X_0, \tilde{A}_0; \hat{\alpha}_0, \hat{B}_0)$.

(vii) Let E be the union of connected components S_0 of S such that $\rho X \cap S_0 \neq \emptyset$. Then $(\rho; E, \tilde{A}|E; \hat{\alpha}|E, \hat{B}|E)$ is the B-envelope of holomorphy of $(X, A; \alpha, B)$. Finally we show that $E = S$; assume $\varphi \in S - E$. Choose K_φ, K_φ compact in X. Let $\hat{B}_{\rho(K_\varphi)}$ be the uniform closure of $\hat{B}|\rho(K_\varphi)$ in $\rho(\rho(K_\varphi))$. Then $S_K = \mathrm{hull}_{\hat{B}}\rho(K_\varphi)$ is compact in its induced weak topology from S and is the maximal ideal space of $\hat{B}_{\rho(K_\varphi)}$. Since E is open and closed, $S_{K_\varphi} \cap E$ is open and closed in S_{K_φ} and is compact. Then, by the Silov idempotent theorem there exists $\hat{g} \in \hat{B}_{\rho(K_\varphi)}$ such that

$$\hat{g}(h) = \begin{cases} 1, & h \in S_{K_\varphi} - (S_{K_\varphi} \cap E) \\ 0, & h \in S_{K_\varphi} \cap E \end{cases}$$

since $\rho(K_\varphi) \subset S_{K_\varphi} \cap E$, $\|\hat{g}\|_{\rho(K_\varphi)} = 0$ while $\hat{g}(\varphi) = 1$. This is absurd. Thus $E = S$. The proof is complete.

DEFINITION 3. A complex manifold (X, A) is a *Stein manifold of bounded type* if it satisfies the following conditions:

(a) For every compact subset K of X, $\hat{K}_B$ is compact,

(b) $B(X)$ separates the points of X,

(c) Every point of X has a local coordinate system consisting of functions in $B(X)$.

EXAMPLES. Any analytic polyhedron in $\boldsymbol{C}^n$, in particular, a polydisc, is a Stein manifold of bounded type, so is an annulus in $\boldsymbol{C}$. A relatively compact domain D in $\boldsymbol{C}$ such that $D = \mathrm{int}\,\bar{D}$ is also a Stein manifold of bounded type.

A punctured open disc in $\boldsymbol{C}$ is not a Stein manifold of bounded type, though it is a Stein manifold. Nor the domains in Examples 1 and 2.

If (X, A) is a Stein manifold of bounded type, it is a Stein manifold. Moreover, by Theorem 4 in Ch. 9 of Cartan [2], $A(X) = \tilde{B}(X)$. Conversely, if (X, A) is a Stein manifold with $A(X) = \tilde{B}(X)$, then it is a Stein manifold of bounded type.

We gather the above discussions in the following theorem.

THEOREM 3. *Let $(X, A; \alpha)$ be a separable Riemann domain with a bounded spread map. Then* (1) *implies* (2) *and* (2) *implies* (3):

(1) *$(X, A; \alpha)$ is a Stein manifold of bounded type.*

(2) *Every nonzero continuous homomorphism of $B(X)$ is the point evaluation of X.*

(3) *$(X, A; \alpha)$ is a domain of bounded holomorphy.*

Furthermore, if $B(X)$ *separates the points of* X *and* $\tilde{B}(X)$ *is stable under differentiation then* (3) *implies* (1).

REMARK. For a domain D of bounded holomorphy, $\tilde{B}(D)$ need not be stable under differentiation; for instance, the domains in Examples 1 and 2.

Example 2 shows that even if $B(D)$ separates the points of D, $\tilde{B}(D)$ is not stable under differentiation, consequently $\tilde{B}(D) \nsubseteq \mathcal{O}(D)$.

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