ATTACHING HUREWICZ FIBRATIONS WITH FIBER PRESERVING MAPS

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When working with fibrations, there are times when standard topological constructions involving identifications are useful. The problem of course, is to show that identifying fibrations in the proper way yields a fibration. This paper establishes a fairly general result concerning attaching Hurewicz fibrations over a fixed base space with a fiber preserving map. This can be applied to obtain many common topological constructions. In particular a theorem of P. Tulley on mapping cylinders is strengthened, which in turn strengthens the main theorems on strong fiber homotopy equivalence and extensions of fibrations obtained by P. Tulley and S. Langston. In addition, these results are applied to obtain a stronger version of Dold's pasting lemma, an important step in the construction of classifying spaces for fibrations.

1. Definitions and notation.

By a space over B we will mean a triple $\xi = (E, p, B)$ where $p: E \to B$ is a map. We will often refer to E as the total space, and B as the base space of ξ. If $A \subset B$ we will let $\xi_A = (E_A, p_A, A) = (p^{-1}(A), p \mid p^{-1}(A), A)$. Given two spaces over B, $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ we will denote by $f: \xi \to \xi'$ a fiber preserving map from ξ to ξ', i.e. a map $f: E \to E'$ such that $p'f = p$. Two such maps $f, g: \xi \to \xi'$ are said to be fiber homotopic denoted $f \sim g: \xi \to \xi'$ if there is a homotopy H from f to g such that for each $t \in I$, $p'H(e, t) = p(e)$. ξ is fiber homotopy equivalent to ξ', denoted $\xi \sim \xi'$, if there are fiber preserving maps $f: \xi \to \xi'$ and $g: \xi' \to \xi$ with fg and gf fiber homotopic to the respective identity maps.

A space over B will be called a Hurewicz fibration, or more simply a fibration if it has the universal covering homotopy property, or equivalently has a lifting function λ. Recall that a lifting function for $\xi = (E, p, B)$ is a section for $(E^I, \tilde{p}, \Omega_p)$, where

$$\Omega_p = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\}$$

and $\tilde{p}(\omega) = (\omega(0), p\omega)$. (i.e. a map $\lambda: \Omega_p \to E^I$ with $\lambda(e, \omega)(0) = e$ and $p\lambda(e, \omega)(t) = \omega(t)$.)

A lifting function is regular if it takes constant paths in the base space to constant paths in the total space, and if a fibration ξ has a regular lifting function, ξ will be called a regular fibration.

DEFINITION (1.1). A slicing function for ξ_A is a map $\mathscr{P}: E_A \times A \to E_A$ such that $\mathscr{P}(e, p(e)) = e$ and $p\mathscr{P}(e, a) = a$.

Note that the existence of a slicing function for ξ_A implies ξ_A is a regular fibration. (A regular lifting function is defined by $\lambda(e, \omega)(t) = \mathscr{P}(e, \omega(t))$.) For our purposes slicing functions are generally easier to work with than lifting functions since the product topology is simpler than the compact open topology of function spaces.

DEFINITION (1.2). A pair of spaces over B (ξ, ξ_0), is a pair of triples $\xi = (E, p, B)$, $\xi_0 = (E_0, p_0, B)$ such that $E_0 \subset E$ and $p_0 = p | E_0$.

If ξ and ξ_0 are both fibrations (ξ, ξ_0) will be called a pair of fibrations. If in addition there is a lifting function $\lambda: \Omega_p \to E^I$ for ξ such that $\lambda | \Omega_{p_0}$ is a lifting function for ξ_0, then (ξ, ξ_0) will be called a fibered pair.

Finally, we will make frequent use of cofibrations.

DEFINITION (1.3). (X, A) is a cofibration if any partial homotopy $\bar{H}: X \times \{0\} \cup A \times I \to Y$ has an extension $H: X \times I \to Y$.

If in addition A is closed we will say (X, A) is a closed cofibration. If X is a Hausdorff space all cofibrations (X, A) are closed.

2. **Preliminary lemmas and the main theorem.**

LEMMA (2.1). (X, A) *is a closed cofibration if and only if there are maps* $D: X \times I \to X$, $\phi: X \to I$ *with the following properties:*
(a) $D(x, 0) = x$ *for all* $x \in X$
(b) $D(a, t) = a$ *for all* $a \in A, t \in I$
(c) $A = \phi^{-1}(1)$ *and* $D(\phi^{-1}(0, 1] \times 1) \subset A$.

The proof, which we will omit, is an exercise in [7], p. 57. The proof is essentially the same as that in [11].

LEMMA (2.2). *If* (ξ, ξ_0) *is a regular fibered pair, then* (E, E_0) *is a closed cofibration if and only if there are maps* $D: E \times I \to E$ *and* $\alpha: E \to I$ *with the following properties:*
(a) $D(e, 0) = e$ *for all* $e \in E$
(b) $D(e, t) = e$ *for all* $e \in E_0, t \in I$
(c) $\alpha^{-1}(1) = E_0$
(d) $D(\alpha^{-1}(0, 1] \times 1) \subset E_0$
(e) $pD(e, t) = p(e)$ *for all* $e \in E, t \in I.$

Note that the only difference between Lemmas (2.1) and (2.2) is part (e), which says that D moves points along fibers.

Proof. Define $D(e, t) = \lambda[D'(e, t), \omega(e, t)](1)$ and $\alpha(e) = \phi(e)$, where D' and ϕ are as in Lemma (2.1), $\omega(e, t)$ is the path given by $\omega(e, t)(s) =$

$pD'(e, (1 - s) \cdot t)$, and λ is a regular lifting function for ξ whose restriction to Ω_{p_0} is a lifting function for ξ_0. Property (b) is a consequence of the regularity of λ, and (d) is a consequence of $\lambda | \Omega_{p_0}$ being a lifting function for ξ_0.

Let $\varDelta B = \{(b, b) \in B \times B\}$ denote the diagonal of B in $B \times B$. The main theorem will be proven for fibrations with paracompact, locally compact base spaces B such that $(B \times B, \varDelta B)$ is a closed cofibration. (ANR's for example satisfy this last condition.) If B is metric then this condition is equivalent to B ULC (uniformly locally contractible). The following lemma in fact concludes that all such spaces are ULC.

LEMMA (2.3).   *If* $(B \times B, \varDelta B)$ *is a closed cofibration then there is a neighborhood* U *of* $\varDelta B$ *and a map* $\sigma: U \to X^I$ *such that*
  (a) $\sigma(x, y)(0) = x$
  (b) $\sigma(x, y)(1) = y$
  (c) $\sigma(x, x)(t) = x$ *for all* $t \in I$.

Proof.   Choose D and ϕ as in Lemma (2.1) and define

$$\sigma(x, y)(t) = \begin{cases} \pi_1 D(x, y, 2t) & 0 \leq t \leq \dfrac{1}{2} \\ \pi_2 D(x, y, 2 - 2t) & \dfrac{1}{2} \leq t \leq 1 \end{cases}$$

where $\pi_i: B \times B \to B$ is projection on the ith coordinate $i = 1, 2$ and $(x, y) \in U = \phi^{-1}(0, 1]$.

The next lemma will allow us a more satisfactory statement of the main theorem.

LEMMA (2.4).   *Let* (ξ, ξ_0) *be a pair of spaces over* B *with* (E, E_0) *a closed cofibration. Then* (ξ, ξ_0) *is a fibered pair if and only if* (ξ, ξ_0) *is a pair of fibrations.*

The proof of this lemma, which we will omit, consists of choosing a lifting function λ_0 for ξ_0 and extending it to a lifting function for ξ. This can be done exactly as in the Lifting Extension Theorem in [1], replacing λ_A and Ω_{p_A} by λ_0 and Ω_{p_0} throughout the proof, and noting that (E, E_0) a closed cofibration implies (Ω_p, Ω_{p_0}) is a closed cofibration. In fact if we let $\pi_1: \Omega_p \to E$ be projection on the first coordinate, then $\Gamma = (\Omega_p, \pi_1, E)$ is a fibration ($\Gamma =$ the pullback by p of the standard path fibration over B), and since $\Omega_{p_0} = \pi_1^{-1}(E_0)$, (Ω_p, Ω_{p_0}) is a closed cofibration by Theorem 12 of [8].

We come now to the main theorem. If (ξ, ξ_0) is a pair of spaces

over B, η a space over B and $f: \xi_0 \to \eta$ a fiber preserving map, we let $\xi \cup_f \eta = (E \cup_f E', q, B)$ where $\xi = (E, p, B)$, $\eta = (E', p', B)$ and q is the map induced by p and p'.

THEOREM (2.5). *Let B be a paracompact, locally compact space such that $(B \times B, \Delta B)$ is a closed cofibration, and (ξ, ξ_0) a pair of fibrations over B with (E, E_0) a closed cofibration. Then for any fibration η over B and $f: \xi_0 \to \eta$, $(\xi \cup_f \eta, \eta)$ is a fibered pair.*

Proof. Since $(B \times B, \Delta B)$ is a closed cofibration, we can define $d: B \times B \to I$ by $d = 1 - \phi$, (ϕ as in Lemma 2.1) with $d^{-1}(0) = \Delta B$. By Lemma (2.4), there is a lifting function λ for ξ whose restriction to Ω_{p_0} is a lifting function for ξ_0. We may assume, in addition, that λ is regular, and that there is a regular lifting function λ' for η. The proof is the same as in [4], using $d: B \times B \to I$ instead of a metric.

Since B is paracompact, the Uniformation Theorem applies (see [4]) and thus we need only show any point has a neighborhood W such that $((\xi \cup_f \eta)_W, \eta_W)$ is a fibered pair. We choose η and σ to be as in Lemma (2.3), and given a point $b_0 \in B$ choose a neighborhood V of b_0 with $V \times V \subset U$. We then have slicing functions ϕ and ψ for ξ_V and η_V respectively defined as follows:

$$\phi(e, x) = \lambda[e, \sigma(p(e), x)](1) \qquad (e, x) \in E_V \times V$$
$$\psi(e', x) = \lambda'[e', \sigma(p'(e'), x)](1) \quad (e', x) \in E'_V \times V \, .$$

It is important to note here that if $e \in E_0$, $\phi(e, x) \in E_0$ for all $x \in V$.

We now let θ be a neighborhood of (b_0, b_0) in U which has the property that $\sigma(x, y)(t) \in V$ for all $(x, y) \in \theta$ and all $t \in I$. There is such a neighborhood since σ is continuous and $\sigma(b_0, b_0)$ is the constant path at b_0. Finally we choose a neighborhood W of b_0 with compact closure such that $\bar{W} \times \bar{W} \subset \theta \subset U$. $\bar{W}$ then the property that for $e \in E_{\bar{W}}$, $e' \in E'_{\bar{W}}$, $x \in \bar{W}$ and $t \in I$, the functions

$$\Phi(e, \sigma(p(e), x)(t)) \quad \text{and} \quad \Psi(e', \sigma(p(e'), x)(t))$$

are defined.

Let D and α be defined for the closed cofibration (E, E_0) as in Lemma (2.2) using the lifting function λ, and $\mu: E_{\bar{W}} \cup E'_{\bar{W}} \to (E \cup_f E')_{\bar{W}}$ be the identification map. Given $D, \alpha, \mu, \phi, \psi, d$ and $\bar{W}$ as above, we now define a map $\bar{\chi}: (E_{\bar{W}} \cup E'_{\bar{W}}) \times \bar{W} \to (E \cup_f E')_{\bar{W}}$ which will induce the slicing function χ for $(\xi \cup_f \eta)_W$. To simplify the notation in the definition of $\bar{\chi}$ we will let $\hat{d} = d(p(e), z)$, and $\hat{\sigma}(t) = \sigma(p(e), z)(t)$.

$\bar{\chi}(e', z) = \mu(\psi(e', z))$ if $e' \in E^L_{\bar{W}}$ and for $e \in E^L_{\bar{W}}$ is given by the following formula:

$$
\bar{\chi}(e, z) = \begin{cases}
\mu\{\phi[D(e, \hat{d}), z]\} & \text{if } 0 \leqq \alpha(e) \leqq 1/8 \\
\mu\{\phi[D(e, 8\alpha(e) \cdot \hat{d}), z]\} & \text{if } 1/8 \leqq \alpha(e) \leqq 1/4 \text{ and} \\
& \quad 0 \leqq 4\alpha(e) \cdot \hat{d} \leqq 1/2 \\
\mu\{f[\phi(D(e, 1), z)]\} & \text{if } 1/8 \leqq \alpha(e) \leqq 1/4 \text{ and} \\
& \quad 1/2 \leqq 4\alpha(e) \cdot \hat{d} \leqq 1 \\
\mu\{\phi[D(e, 2\hat{d}), z]\} & \text{if } 1/4 \leqq \alpha(e) \leqq 1/2 \text{ and } 0 \leqq \hat{d} \leqq 1/2 \\
\mu\{\psi[f(\phi[D(e, 1), \hat{\sigma}(2 - 4\alpha(e) + (4\alpha(e) - 1)/2\hat{d})]), z]\} & \text{if} \\
& \quad 1/4 \leqq \alpha(e) \leqq 1/2 \text{ and } 1/2 \leqq \hat{d} \leqq 1 \\
\mu\{\phi[D(e, \hat{d}/(1 - \alpha(e))), z]\} & \text{if } 1/2 \leqq \alpha(e) \leqq 1, \alpha(e) \neq 1 \text{ and} \\
& \quad 0 \leqq \hat{d} \leqq 1 - \alpha(e) \\
\mu\{\psi[f(\phi[D(e, 1), \hat{\sigma}((1 - \alpha(e))/\hat{d})]), z]\} & \text{if } 1/2 \leqq \alpha(e) \leqq 1, \\
& \quad \hat{d} \neq 0 \text{ and } 1 - \alpha(e) \leqq \hat{d} \leqq 1 \\
\mu(e) & \text{if } \alpha(e) = 1 \text{ and } \hat{d} = 0 .
\end{cases}
$$

The function $\bar{\chi}: (E_{\bar{W}} \cup E^L_{\bar{W}}) \times \bar{W} \to (E \cup_f E')_{\bar{W}}$ is continuous, and has the property that for $e \in E_0$,

$$
\begin{aligned}
\bar{\chi}(e, z) &= \mu\{\psi[f(\phi(e, p(e)), z]\} \\
&= \mu\{\psi[f(e), z]\} \\
&= \bar{\chi}(f(e), z) .
\end{aligned}
$$

Since $\bar{W}$ is compact, $(E \cup_f E')_{\bar{W}} \times \bar{W}$ has the identification topology induced by the map $\mu \times 1: (E_{\bar{W}} \cup E^L_{\bar{W}}) \times \bar{W} \to (E \cup_f E')_{\bar{W}} \times \bar{W}$ (see Lemma 4 of [10]). Thus since $\bar{\chi}(e, z) = \bar{\chi}(f(e), z)$ for $e \in E_0$, χ induces a map $\chi: (E \cup_f E')_{\bar{W}} \times \bar{W} \to (E \cup_f E')_{\bar{W}}$. To complete the proof we need only check that χ is a slicing function for $(\xi \cup_f \eta)_{\bar{W}}$. We leave these details to the reader. Also we have claimed that $(\xi \cup_f \eta, \eta)$ is a fibered pair. To see this note that the slicing function χ when restricted to $E'_{\bar{W}} \times \bar{W}$ is a slicing function for $\eta_{\bar{W}}$. Thus the local lifting functions defined by these slicing functions will be lifting functions of pairs, and the Uniformization Theorem will give us a lifting function for $\xi \cup_f \eta$ whose restriction is a lifting function for η.

Before considering applications of Theorem (2.5), the natural question arises as to whether or not the fiber homotopy type of the fibrations we obtain depends only on the fiber homotopy class of the attaching map. An affirmative answer is provided by the next theorem.

DEFINITION (2.6). We will say a pair of spaces (ξ, ξ_0) over B is a closed cofibration if we can define maps $D: E \times I \to E$ and $\alpha: E \to I$

satisfying the properties given in Lemma (2.2).

Lemma (2.2) then shows that a pair of fibrations (ξ, ξ_0) is a closed cofibration if and only if (E, E_0) is a closed cofibration.

THEOREM (2.7). *Let* (ξ, ξ_0) *be a closed cofibration. Then if* $f \sim g: \xi_0 \rightarrow \eta$, $\xi \cup_f \eta$ *is fiber homotopy equivalent to* $\xi \cup_g \eta$.

Note that we do not assume ξ *or* ξ_0 *are fibrations. Also note that all maps and homotopies involved in the proof of the theorem leave* η *fixed. One could state this as a relative theorem, making the proper definitions of relative fiber homotopies etc.*

Proof. Let $G: E_0 \times I \rightarrow E'$ be a fiber homotopy with $G(e, 0) = f(e)$, $G(e, 1) = g(e)$, and let $D: E \times I \rightarrow E$, $\alpha: E \rightarrow I$ be as in Definition (2.7). Let $\mu: E \cup E' \rightarrow E \cup_f E'$ and $\nu: E \cup E' \rightarrow E \cup_g E'$ denote the identification maps. We define $\bar{h}: E \cup E' \rightarrow E \cup_g E'$, and $\bar{h}': E \cup E' \rightarrow E \cup_f E'$ as follows:

$$
\bar{h}(e) = \begin{cases} \nu[D(e, 2\alpha(e))] & \text{if } e \in E \text{ and } 0 \leq \alpha(e) \leq 1/2 \\ \nu[G(D(e, 1), 2 - 2\alpha(e))] & \text{if } e \in E \text{ and } 1/2 \leq \alpha(e) \leq 1 \\ \nu(e) & \text{if } e \in E' , \end{cases}
$$

$$
\bar{h}'(e) = \begin{cases} \mu[D(e, 2\alpha(e))] & \text{if } e \in E \text{ and } 0 \leq \alpha(e) \leq 1/2 \\ \mu[G(D(e, 1), 2\alpha(e) - 1] & \text{if } e \in E \text{ and } 1/2 \leq \alpha(e) \leq 1 \\ \mu(e) & \text{if } e \in E' . \end{cases}
$$

$\bar{h}$ and $\bar{h}'$ are continuous and induce fiber preserving maps $h: E \cup_f E' \rightarrow E \cup_g E'$ and $h': E \cup_g E' \rightarrow E \cup_f E'$ respectively.

We define $\bar{H}: (E \cup E') \times I \rightarrow E \cup_f E'$ and $\bar{K}: (E \cup E') \times I \rightarrow E \cup_g E'$ which induce fiber preserving homotopies $H:(E \cup_f E') \times I \rightarrow E \cup_f E'$ from the identity map to $h'h$ and $K: (E \cup_g E') \times I \rightarrow E \cup_g E'$ from the identity map to hh' by the following formulas:

$$
\bar{H}(e, t) = \begin{cases} \mu[D(D(e, 2\alpha(e) \cdot t), 2\alpha(D(e, 2\alpha(e) \cdot t)) \cdot t)] \\ \qquad \text{if } e \in E, 0 \leq \alpha(e) \cdot t \leq 1/2 \text{ and} \\ \qquad 0 \leq \alpha(D(e, 2\alpha(e) \cdot t)) \cdot t \leq 1/2 \\ \mu[G\{D[D(e, 2\alpha(e) \cdot t), 1], 2\alpha(D(e, 2\alpha(e) \cdot t)) \cdot t - 1] \\ \qquad \text{if } e \in E, 0 \leq \alpha(e) \cdot t \leq 1/2 \text{ and} \\ \qquad 1/2 \leq \alpha(D(e, 2\alpha(e) \cdot t \leq 1 \\ \mu[G(D(e, 1), (2 - 2\alpha(e)) \cdot t))] \\ \qquad \text{if } e \in E \text{ and } 1/2 \leq \alpha(e) \cdot t \leq 1 \\ \mu(e) \qquad \text{if } e \in E' , \end{cases}
$$

$$\bar{K}(e, t) = \begin{cases} \nu[D(D(e, 2\alpha(e) \cdot t), 2\alpha(D(e, 2\alpha(e) \cdot t)) \cdot t)] \quad \text{if} \\ \qquad e \in E, \ 0 \leqq \alpha(e) \cdot t \leqq 1/2 \quad \text{and} \\ \qquad 0 \leqq \alpha(D(e, 2\alpha(e) \cdot t)) \cdot t \leqq 1/2 \\ \nu[G(D(D(e, 2\alpha(e) \cdot t), 1), 2 - 2\alpha(D(e, 2\alpha(e) \cdot t)) \cdot t)] \quad \text{if} \\ \qquad e \in E, \ 0 \leqq \alpha(e) \cdot t \leqq 1/2 \quad \text{and} \\ \qquad 1/2 \leqq \alpha(D(e, 2\alpha(e) \cdot t)) \cdot t \leqq 1 \\ \nu[G(D(e, 1), \beta(e, t))] \text{ if } e \in E \quad \text{and} \quad 1/2 \leqq \alpha(e) \cdot t \leqq 1 \\ \nu(e) \quad \text{if} \quad e \in E'. \end{cases}$$

$\beta(e, t) = \min\{4\alpha(e) \cdot t - 2\alpha(e) \cdot t^2 - t, 1\}$ in the preceding definition of $\bar{K}$.

3. Applications.

Given $\xi = (E, p, B)$, $\eta = (E', p', B)$ and $f: \xi \to \eta$ we will denote by $\mu_f = (M_f, q, B)$ the space over B with total space the mapping cylinder of f, and q the map induced by p' and $\bar{p}: E \times I \to B$ where $\bar{p}(e, t) = p(e)$. The following theorem is an immediate corollary to Theorem (2.5).

THEOREM (3.1). *Let B be a paracompact, locally compact space with the property that $(B \times B, \Delta B)$ is a closed cofibration. Then if ξ and η are fibrations over B, and if $f: \xi \to \eta$ is a fiber preserving map, μ_f is a fibration.*

The notion of strong fiber homotopy equivalence (due to P. Tulley) is studied and used extensively by P. Tulley in [9] and S. Langston in [6].

DEFINITION (3.2). Two fibrations $\xi = (E, p, B)$, and $\eta = (E', p', B)$ have the same strong fiber homotopy type denoted $\xi \sim_s \eta$ if there is a fibration $\Gamma = (E^*, p^*, B \times I)$ with $\Gamma_{B \times \{0\}} = \xi$ and $\Gamma_{B \times \{1\}} = \eta$.

In [6] Langston proved that fiber homotopy equivalence is the same as strong fiber homotopy equivalence when E and E' are separable metric ANR's. In [9] Tulley proved that if $f: \xi \to \eta$ is a fiber homotopy equivalence of fibrations, and if μ_f is a fibration then $\xi \sim_s \eta$. The result in [9] however, assumed E was compact to show μ_f is a fibration. By Theorem (3.1) and Theorem 11 in [9] we then have the following result.

THEOREM (3.3). *If B is paracompact, locally compact and $(B \times B, \Delta B)$ a closed cofibration, then for any two fibrations ξ and η, $\xi \sim \eta$ if and only if $\xi \sim_s \eta$.*

In [6] Langston used the relationship between strong fiber homotopy equivalence and fiber homotopy equivalence to consider the problem of extending a fibration.

DEFINITION (3.4).   If $A \subset B$ and $\xi = (E, p, A)$ is a fibration, then a fibration η is an extension of ξ if $\eta_A = \xi$.

Langston in [6] proved that if (B, A) is a finite CW pair, and if two fibrations ξ and ξ' over A with ANR total spaces are fiber homotopy equivalent, then ξ can be extended to B if and only if ξ' extends to B. The next theorem will generalize this result. We prove a lemma first to give a better statement of the theorem.

LEMMA (3.5).   *If (X, A) and $(X \times X, \Delta X)$ are closed cofibrations, A metric, then $(A \times A, \Delta A)$ is a closed cofibration.*

Proof.   Let $D: X \times I \to X, \phi: X \to I$ be associated with the pair (X, A) and $\bar{D}: X \times X \times I \to X \times X, \bar{\phi}: X \times X \to I$ with the pair $(X \times X, \Delta X)$ as in Lemma (2.1). Let $V = \phi^{-1}(0, 1] \subset X$ and $r: V \to A$ be the retraction $r(x) = D(x, 1)$. Let $U = \bar{\phi}^{-1}(0, 1) \subset X \times X$, and define $\sigma: U \to X^I$ as in Lemma (2.3) using $\bar{D}$ and $\bar{\phi}$.

Let W be a relatively open set in $A \times A$ with the property that $\sigma^{-1}(V) \cap A \times A \supset \bar{W} \supset W \supset \Delta A$. ($\bar{W}$ = closure of W.) We then have $\sigma(x, y)(t) \in V$ for all $(x, y) \in \bar{W}$. Choose $\alpha: A \times A \to I$ such that $\alpha^{-1}(1) = \Delta A$ and $\alpha(A \times A - W) = 0$. We define $\hat{D}: A \times A \times I \to A \times A$ and $\hat{\phi}: A \times A \to I$ as follows:

$$\hat{D}(x, y, t) = \begin{cases} (x, y) & \text{if } (x, y) \in A \times A - W \\ (r\sigma(x, y)(\alpha(x, y) \cdot t), r\sigma(x, y)(1 - \alpha(x, y) \cdot t)) \\ & \text{if } 0 \leqq \alpha(x, y) \leqq 1/2 \text{ and } (x, y) \in \bar{W} \\ (r\sigma(x, y)(1/2 \cdot t), r\sigma(x, y)(1 - 1/2 \cdot t)) \\ & \text{if } 1/2 \leqq \alpha(x, y) \leqq 1 \text{ and } (x, y) \in \bar{W} , \end{cases}$$

$$\hat{\phi}(x, y) = \begin{cases} 0 & \text{if } 0 \leqq \alpha(x, y) \leqq 1/2 \\ 2\alpha(x, y) - 1 & \text{if } 1/2 \leqq \alpha(x, y) \leqq 1 . \end{cases}$$

Using $\hat{D}$ and $\hat{\phi}$, Lemma (2.1) shows that $(A \times A, \Delta A)$ is a closed cofibration.

THEOREM (3.6).   *Let B be a metric space such that $(B \times B, \Delta B)$ is a cofibration, and A a locally compact subset of B with (B, A) a cofibration. Then if ξ and η are fiber homotopy equivalent fibrations over A, ξ can be extended to B if and only if η extends to B.*

Proof. $\xi = (E, p, A)$ and $\eta = (E', p', A)$ are strongly fiber homotopy equivalent by Theorem (3.3), therefore there is a fibration $\Gamma =$

$(E^*, p^*, A \times I)$ with $\Gamma_{A \times \{0\}} = \xi$ and $\Gamma_{A \times \{1\}} = \eta$. If ξ has an extension $\bar{\xi} = (\bar{E}, \bar{p}, B)$, consider the space over $B \times \{0\} \cup A \times I$

$$\bar{\xi} \cup \Gamma = (\bar{E} \cup E^*, \bar{p} \cup p^*, B \times \{0\} \cup A \times I) .$$

Since $(A \times I, A \times \{0\})$ and $(B \times \{0\}, A \times \{0\})$ are closed cofibrations, Theorem (4.2) of [1] applies, and $\bar{\xi} \cup \Gamma$ is a fibration. Let $r: B \times I \to B \times \{0\} \cup A \times I$ be a retraction. Then $r^*(\bar{\xi} \cup \Gamma)$, the pullback of $\xi \cup \Gamma$ by r, is a fibration, and the desired extension of η is given by $r^*(\bar{\xi} \cup \Gamma)_{B \times \{1\}}$.

REMARK. Note that the extension of η given by this theorem is fiber homotopy equivalent to the extension of ξ that we started with.

We now apply Theorem (3.6) and Theorem (4.2) of [1] to prove a stronger form of Dold's pasting lemma. (See §6.4, of [2].)

THEOREM (3.7). *Let X be a metric space of the form $X = X_1 \cup X_2$ with $X_1 \cap X_2$ locally compact, and $(X_i, X_1 \cap X_2)$ and $(X_i \times X_i, \Delta X_i)$ cofibrations $i = 1, 2$. Then if $\xi^1 = (E_1, p_1, X_1)$ and $\xi^2 = (E_2, p_2, X_2)$ are fibrations such that $\xi^1_{X_1 \cap X_2} \sim \xi^2_{X_1 \cap X_2}$, there is a fibration $\eta = (E, p, X_1 \cup X_2)$ with $\eta_{X_1} = \xi^1$ and $\eta_{X_2} \sim \xi^2$.*

Proof. $\xi^1_{X_1 \cap X_2} \sim \xi^2_{X_1 \cap X_2}$ and $\xi^2_{X_1 \cap X_2}$ extends to ξ^2 over X_2, therefore by Theorem (3.6), there is an extension $\bar{\xi} = (\bar{E}, \bar{p}, X_2)$ of $\xi^1_{X_1 \cap X_2}$ to X_2 which is fiber homotopy equivalent to ξ^2. We now let

$$\eta = (\bar{E} \cup \bar{E}_1, \bar{p} \cup p_1, X_1 \cup X_2) .$$

η is fibration by Theorem (4.2) of [1] and $\eta_{X_1} = \xi^1$, $\eta_{X_2} \sim \xi^2$ by construction.

As a final application of Theorem (2.5) we consider the problem of attaching a disk bundle to a fibration by a fiber preserving map on the associated sphere boundle. Ibisch in [5] studied this problem for weak fibrations i.e. triples with the weak covering homotopy property. If ξ is a disk bundle over B, B as in Theorem (2.5), it is clear that at least locally, the total space of the associated sphere bundle ξ_0 is a closed cofibration in the total space of ξ. Therefore given a fibration η over B and a fiber preserving map $f: \xi_0 \to \eta$, we can apply Theorem (2.5) locally to get $\xi \cup_f \eta$ is a fibration.

REMARK. Using Theorem 2 of [3] one can show in fact that the inclusion of the total space of ξ_0 in the total space of ξ is a closed cofibration.

By a different method we can eliminate the condition $(B \times B, \Delta B)$ is a closed cofibration. Let

$$\xi = (E_1 \cup E_2,\ p,\ B),\ \xi^1 = (E_1,\ p_1,\ B),\ \xi^2 = (E_2,\ p_2,\ B)$$

and

$$\xi^0 = (E_1 \cap E_2,\ p_0,\ B) = (E_0,\ p_0,\ B)$$

with $p_i = p \,|\, E_i$.

LEMMA (3.8). *If* (E_i, E_0) $i = 1, 2$ *are closed cofibrations and* ξ^0, ξ^1, ξ^2 *are fibrations, then* ξ *is a fibration.*

Proof. Choose a lifting function λ_0 for ξ^0. By Lemma (2.4) we can find lifting functions $\lambda_i \colon \Omega_{p_i} \to E_i^I$ $i = 1, 2$, for ξ^2 whose restrictions to Ω_{p_0} are λ_0. A lifting function for ξ is then defined by

$$\lambda(e, \omega) = \begin{cases} \lambda_1(e, \omega) & \text{if} \quad (e, \omega) \in \Omega_{p_1} \\ \lambda_2(e, \omega) & \text{if} \quad (e, \omega) \in \Omega_{p_2} . \end{cases}$$

Now let $\xi = (E, p, B)$ be a disk bundle with $\dot{\xi} = (\dot{E}, \dot{p}, B)$ the boundary sphere bundle, and $\eta = (E', p', B)$ a fibration.

THEOREM (3.9). *Let* B *be a paracompact, locally compact space and* $f \colon \dot{\xi} \to \eta$. *Then* $\xi \cup_f \eta$ *is a fibration.*

Proof. Consider the mapping cylinder $\mu_f = (M_f, \bar{q}, B)$. Given $b_0 \in B$, we can choose a neighborhood W of b_0 with compact closure, so that $\dot{\xi}$ is trivial over $\bar{W}$. $E_{\bar{W}}$ is then compact, therefore we can apply. Theorem (8) in [9] to prove μ_f is a fibration over $\bar{W}$. μ_f, ξ, and $\dot{\xi}$ is thus a fibration by the Uniformization Theorem. Letting $\| \cdot \|$ denote the norm induced by a Riemannian metric on ξ, we define $E_i \subset E \cup_f E'$ as follows:

$$E_1 = \{ e \in E \mid \|e\| \geqq 1/2 \} \cup_f E'$$
$$E_2 = \{ e \in E \mid \|e\| \leqq 1/2 \} .$$

Then $(E_1, q \,|\, E_1, B), (E_2, q \,|\, E_2, B)$ and $(E_1 \cap E_2, q \,|\, E_1 \cap E_2, B)$ are fiberwise homeomorphic to μ_f, ξ, and $\dot{\xi}$ respectively and are thus fibrations. $(E_i, E_1 \cap E_2)$ is a closed cofibration $i = 1, 2$. Therefore Lemma (3.8) applies, and $\xi \cup_f \eta$ is a fibration.

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