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ERGODICITY IN VON NEUMANN ALGEBRAS

CHARLES RADIN

We investigate the ergodicity of elements of a von Neumann algebra $\mathfrak{A}$ under the action of an arbitrary cyclic group of inner *-automorphisms of $\mathfrak{A}$. A simple corollary of our results is the following characterization: A von Neumann algebra $\mathfrak{A}$ is finite if and only if for each $A \in \mathfrak{A}$ and inner *-automorphism α of $\mathfrak{A}$, there exists $\bar{A} \in \mathfrak{A}$ such that $1/N \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N\to\infty]{} \bar{A}$ in the weak operator topology.

1. **Introduction.** Our purpose is to explore in a new direction the ergodic theory of von Neumann algebras presented by Kovács and Szücs [2]. In [2] the essential contribution was the introduction of a certain restriction (called G-finiteness) on a group of *-automorphisms of a von Neumann algebra, fashioned so that all elements of the algebra behave ergodicly with respect to the group. Instead we consider the action of a natural class of (cyclic) groups of *-automorphisms, namely the inner ones, and investigate which elements of the algebra behave ergodicly with respect to all such groups.

2. **Behavior of infinite projections.** From the ergodic theory developed in [2], we note the following simple consequence.

THEOREM 0. (Kovács and Szücs). *Let $\mathfrak{A}$ be a finite von Neumann algebra. For each $A \in \mathfrak{A}$ and each inner *-automorphism α of $\mathfrak{A}$, there exists $\bar{A} \in \mathfrak{A}$ such that $1/N \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N\to\infty]{} \bar{A}$ in the strong operator topology.*

Our first result is a complement to this and provides a new characterization of finiteness for von Neumann algebras.

THEOREM 1. *Let $\mathfrak{A}$ be a von Neumann algebra. For each nonzero infinite projection $P \in \mathfrak{A}$ there exists an infinite projection $\theta \in \mathfrak{A}$, $\theta \leqq P$, and a unitary $U \in \mathfrak{A}$, such that $1/N \sum_{n=0}^{N-1} U^n \theta U^{-n}$ does not converge in the weak operator topology.*

First we need the following lemma.

LEMMA. *There exists a nonzero properly infinite projection $P' \leqq P$.*

Proof. Let S be the set of all central projections E of $\mathfrak{A}$ such that EP is finite. $0 \in S$ so S is not empty. Let $\{E_\alpha\}$ be an orthogonal family of elements of S. If $\sum_\alpha E_\alpha P \sim Q \leqq \sum_\alpha E_\alpha P$ (where $\sim$ is the usual equivalence relation for projections in $\mathfrak{A}$), then $E_\alpha P \sim E_\alpha Q \leqq E_\alpha P$ so that $E_\alpha Q = E_\alpha P$ and therefore $Q \geqq \sum_\alpha E_\alpha Q = \sum_\alpha E_\alpha P$. Therefore, $Q = \sum_\alpha E_\alpha P$ and $\sum_\alpha E_\alpha P$ is finite. It follows easily that there exists a (unique) maximal element F in S. From [1, III.2.3.5] it follows that $(I-F)P$ is nonzero and infinite. Assume it is not properly infinite. Then from [1, III.2.5.9] there exists a central projection G such that $0 \neq G(I-F)P$ is finite. But then from [1, III.2.3.5] $F < F + G(I-F) \in S$, which contradiction proves our lemma with $P' \equiv (I-F)P$.

Proof of Theorem 1. From [1, III.8.6.2] there exists a set $\{P_n \mid n \in \boldsymbol{Z}\}$ of nonzero projections $P_n \in \mathfrak{A}$ such that $P_n P_m = \delta_{n,m} P_n$ and $P_n \sim P_m$ for all $m, n \in \boldsymbol{Z}$, and such that $\sum_{|n| \leqq m} P_n \xrightarrow[m \to \infty]{} P'$ in the strong operator topology. Therefore, there exist $V_n \in \mathfrak{A}$ such that $V_n^* V_n = P_n$ and $V_n V_n^* = P_{n+1}$ for all $n \in \boldsymbol{Z}$, so that $P_{n+1} V_n = V_n P_n$ and $P_n V_n^* = V_n^* P_{n+1}$ for all $n \in \boldsymbol{Z}$. Define for each $f \in \mathscr{H}$ (the Hilbert space of definition of $\mathfrak{A}$),

$$Uf = (\text{norm} \lim_{m \to \infty} \sum_{|n| \leqq m} V_n P_n f) + (I - P')f \,,$$

where the limit exists since $\| V_n P_n f \| = \| P_n f \|$ and $V_n P_n f = P_{n+1} V_n f$ so that $\{V_n P_n f \mid n \in \boldsymbol{Z}\}$ are pairwise orthogonal and

$$\sum_{|n| \leqq m} \| V_n P_n f \|^2 = \sum_{|n| \leqq m} \| P_n f \|^2 \leqq \| P' f \|^2 \,.$$

In fact U is clearly a linear and norm preserving surjection, and therefore unitary. Now since

$$\Big(\sum_{|k| \leqq l} V_k P_k\Big) \text{norm} \lim_{m \to \infty} \sum_{|n| \leqq m} P_n f = \sum_{|n| \leqq l} V_n P_n f$$

it follows that $U_l \equiv I - P' + \sum_{|k| \leqq l} V_k P_k$ has U as a strong operator limit as $l \to \infty$. Therefore, $U \in \mathfrak{A}$. It also follows that $U P_n U^{-1} = P_{n+1}$ for all $n \in \boldsymbol{Z}$, and so by induction $U^m P_n U^{-m} = P_{n+m}$ for all $m, n \in \boldsymbol{Z}$. Now define $g \colon \boldsymbol{N} \to \{0, 1\}$ by

$$g(n) = \begin{cases} 1 & \text{if} \quad 3^{2m} \leqq n < 3^{2m+1} \quad \text{for some} \quad m \in \boldsymbol{N} \\ 0 & \text{if} \quad 3^{2m+1} \leqq n < 3^{2m+2} \quad \text{for some} \quad m \in \boldsymbol{N} \,. \end{cases}$$

Then define θ as the strong operator limit as

$$K \to -\infty \quad \text{of} \quad \sum_{m=K}^{0} g(-m) P_m \,,$$

and let ψ be a unit vector in $P_0 \mathscr{H}$. Now consider

$$\left\langle \psi, 1/N \sum_{n=0}^{N-1} U^n \theta U^{-n} \psi \right\rangle = 1/N \sum_{n=0}^{N-1} \left\langle \psi, U^n \theta U^{-n} P_0 \psi \right\rangle$$
$$= 1/N \sum_{n=0}^{N-1} \sum_{m=-\infty}^{0} g(-m) \left\langle \psi, P_{n+m} P_0 \psi \right\rangle$$
$$= 1/N \sum_{n=0}^{N-1} g(n) .$$

It is easy to see that for all $M \in N$, $1/3^{2M+1} \sum_{n=0}^{3^{2M+1}-1} g(n) \geqq 2/3$ yet $1/3^{2M+2} \sum_{n=0}^{3^{2m+2}-1} g(n) \leqq 1/3$, and the theorem is proven.

Using Theorem 0, we have immediately,

COROLLARY 1 (resp.2). *A von Neumann algebra $\mathfrak{A}$ is finite if and only if for each $A \in \mathfrak{A}$ and inner *-automorphism α of $\mathfrak{A}$, there exists $\bar{A} \in \mathfrak{A}$ such that $1/N \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N\to\infty]{} \bar{A}$ in the weak (resp. strong) operator topology.*

3. Finite elements. Theorem 1 raises the question of the ergodic behavior, under arbitrary inner *-automorphisms, of "finite elements" of infinite von Neumann algebras. The following theorem gives some information in this direction.

THEOREM 2. *Let $\mathfrak{A}$ be a von Neumann algebra and τ a faithful normal semi-finite trace on $\mathfrak{A}^+$ invariant under the *-automorphism α of $\mathfrak{A}$. Then for each $A \in \mathfrak{A}$ such that $\tau(A^*A) < \infty$, there exists $\bar{A} \in \mathfrak{A}$ such that $1/N \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N\to\infty]{} \bar{A}$ in the strong operator topology.*

Proof. First we define the following (standard) objects: see e.g. [1, I.6.2.2]

$$\| \ \|_2 : A \in \mathfrak{A} \longrightarrow [\tau(A^*A)]^{1/2}$$
$$\mathscr{N} = \{A \in \mathfrak{A} \mid \| A \|_2 < \infty\} .$$

Let L_2 be the abstract completion of $\mathscr{N}$ in the norm $\| \ \|_2$, and extend $\| \ \|_2$ to L_2 in the usual way. Let i be the isometric embedding of $\mathscr{N}$ into L_2. L_2 is a Hilbert space with the obvious addition and scalar multiplication, and inner product $<, >$ defined as the extension to $L_2 \times L_2$ of

$$\tau : A \times B \in \mathscr{N} \times \mathscr{N} \longrightarrow \tau(A^*B) .$$

We note the simple inequalities

$$\| AB \|_2 \leqq \| A \| \, \| B \|_2 \qquad \text{for all } B \in \mathscr{N}, \ A \in \mathfrak{A}$$
$$\| AB \|_2 \leqq \| A \|_2 \, \| B \| \qquad \text{for all } B \in \mathscr{N}, \ B \in \mathfrak{A} .$$

We then define the C^*-representation π of $\mathfrak{A}$ on L_2 by

$$\pi(A)i(B) \equiv i(AB)$$

and noting that $\|\pi(A)i(B)\|_2 = \|AB\|_2 \leqq \|A\|\,\|B\|_2$ so that $\pi(A)$ extends uniquely to L_2 by continuity. It is easy to see that π is faithful and normal and that

$$U\colon i(B) \longrightarrow i(\alpha[B]) \qquad \text{for } B \in \mathscr{N}$$

extends to a unitary operator on L_2. Defining, for $B \in \mathfrak{A}$,

$$B_N = \frac{1}{N}\sum_{n=0}^{N-1} \alpha^n(B), \quad \text{we know by von Neumann's}$$

mean ergodic theorem that for each $A \in \mathscr{N}$, $i(A_N)$ is $\|\ \|_2$-Cauchy. Define for each $B \in \mathscr{N}$,

$$D_A\colon i(B) \longrightarrow \text{norm} \lim_{N\to\infty} \pi(A_N)i(B)$$

which limit exists since

$$\|\pi(A_N - A_M)i(B)\|_2 \leqq \|A_N - A_M\|_2 \|B\| \,.$$

D_A is obviously linear. Furthermore,

$$\|D_A i(B)\|_2 = \lim_{N\to\infty} \|\pi(A_N)i(B)\|_2 \leqq \|A\|\,\|B\|_2$$

so D_A extends uniquely to a bounded operator on L_2 by continuity. It is easy to see that $\pi(A_N)$ converges to D_A in the strong operator topology. Since π is normal, $\pi(\mathfrak{A})$ is strong operator closed [1, I.4.3.2] so there exists $\bar{A} \in \mathfrak{A}$ such that $D_A = \pi(\bar{A})$. Since π is faithful, $A_N \xrightarrow[N\to\infty]{} \bar{A}$ in the strong operator topology [1, I.4.3.1].

COROLLARY 1. *Let $\mathfrak{A}$ be a countably decomposable von Neumann algebra. For each finite projection $P \in \mathfrak{A}$ and inner *-automorphism α of $\mathfrak{A}$, there exists $\bar{P} \in \mathfrak{A}$ such that*

$$\frac{1}{N}\sum_{M=0}^{N-1} \alpha^n(P) \xrightarrow[N\to\infty]{} \bar{P} \quad \textit{in the strong operator topology}\,.$$

Proof. Let

$$A \in \mathfrak{A} \longrightarrow A_1 \oplus A_2 \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$$

be the canonical decomposition of $\mathfrak{A}$ into its countably decomposable semi-finite and purely infinite components. From [1, I.6.7.9] we know that any finite countably decomposable von Neumann algebra has a faithful, normal, tracial state. Inserting this fact into the proof of

[3, 2.5.3], we see that there exists a countable faithful family $\{\tau_n \mid n \in N\}$ of normal semi-finite traces on $\mathfrak{A}_1^+$ with pairwise orthogonal supports such that $\tau_n(P_1) < \infty$ for all $n \in N$. Define

$$\tau' = \sum_{n=0}^{\infty} \tau_n/[\tau_n(P_1) + 2]^n$$

on $\mathfrak{A}_1^+$; it is faithful, normal and semi-finite. Since α is also inner for $\mathfrak{A}_1$ and therefore leaves τ' invariant, we may apply Theorem 2 to $\mathfrak{A}_1$. Since $P_2 = 0$ from [1, III.2.4.8], we are finished.

In the countably decomposable case, Theorem 2 gives us an essentially different proof of Theorem 0, namely

COROLLARY 2. *Let $\mathfrak{A}$ be a finite countably decomposable von Neumann algebra. For each $A \in \mathfrak{A}$ and inner $*$-automorphism α of $\mathfrak{A}$, there exists $\bar{A} \in \mathfrak{A}$ such that*

$$\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N \to \infty]{} \bar{A} \quad \textit{in the strong operator topology}\,.$$

Proof. Just combine the existence of a faithful finite normal trace on $\mathfrak{A}^+$ [1, I.6.7.9] with Theorem 2.

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PRINCETON UNIVERSITY