PACIFIC JOURNAL OF MATHEMATICS
Vol. 48, No. 2, 1973

GROUPS OF ISOMETRIES ON ORLICZ SPACES

JEROME A. GOLDSTEIN

If X is an Orlicz space of functions on an atomic measure space, then, roughly, the only strongly continuous groups of isometries on X are trivial, unless X is a Hilbert space. Hilbert space is thus characterized among the Orlicz spaces on an atomic measure space by its great abundance of strongly continuous isometric groups.

1. **Introduction.** Let X be a real or complex Orlicz space of functions on an atomic measure space; an additional (not very restrictive) condition will be imposed on X which implies in particular that $X \neq L^\infty$. If X is a Hilbert space, there are numerous strongly continuous one parameter groups of isometries on X, according to a classical theorem of M. H. Stone; namely, each skew-adjoint operator on X generates such a group. We shall show that this property characterizes the Hilbert spaces among the Orlicz spaces under consideration on an atomic measure space. Our main result is, roughly, if $\{T_t: t \in \boldsymbol{R} = (-\infty, \infty)\}$ is a strongly continuous (or (C_0)) group of linear isometries on X and if X is not a Hilbert space, then for each real t, T_t has the following form: $(T_t f)(w) = \exp\{i \cdot t g(w)\} f(w)$ for $f \in X$ and $w \in \Omega$ if X is complex, where g is a real-valued function on Ω; or $T_t = I$ ($=$ the identity operator on X) if X is real.

Section 2 contains some preliminaries, including a discussion of duality maps for Orlicz spaces. The main result is stated and proved in § 3. Section 4 contains some complements and examples, including a proof of the main theorem for finite dimensional L^∞ spaces.

The present paper has several points of contact with Lumer's paper [9], which we became aware of shortly after the present paper was submitted for publication.

2. **Preliminaries.** For general facts about Orlicz spaces, convenient references are [6], [17], [11], [12], and [13]. Let (Ω, Σ, μ) be a measure space and let L^Φ be a real or complex Orlicz space on it. Let Ψ be the convex function complementary to Φ in the sense of Young. We normalize Φ, Ψ so that $\Phi(1) + \Psi(1) = 1$; this can always be done according to [17, p. 173]. Then norm in L^Φ is defined by

$$\| f \|_\Phi = \inf \left\{ k > 0 \colon \int_\Omega \Phi(k^{-1} | f |) d\mu \leqq \Phi(1) \right\},$$

and similarly for the norm $\| \cdot \|_\Psi$ in L^Ψ. Every $\phi \in L^\Psi$ defines a bounded linear functional on L^Φ by means of the map $f \to \int_\Omega f \phi d\mu$; moreover, the norm of this functional is $\|\phi\|_\Psi$.

We shall only consider atomic measure spaces, and we shall suppose without loss of generality that each atom has finite positive measure. The assumption that no atom has zero measure enables us to view members of L^Φ as functions rather than as equivalence classes of such. The assumption that no atom has infinite measure means simply that the measure space has the finite subset property. This assumption in no way restricts the generality since each $f \in L^\Phi$ necessarily vanishes on all atoms of infinite measure, and so we could delete from Ω its atoms of infinite measure without changing L^Φ, as long as $\Phi(x) > 0$ for $x > 0$.

We shall use the terminology of Hille and Phillips [5], [4] concerning semigroups of linear operators. Let Y be a real or complex Banach space with dual space Y^*. For $f \in Y$, $\phi \in Y^*$, the value of ϕ at f will be denoted by $\langle f, \phi\rangle$. For $f \in Y$ let $\mathscr{J}f$ be the (nonempty) set of all $\phi \in Y^*$ such that $\|\phi\| = \|f\|$ and $\langle f, \phi\rangle = \|f\|^2$. A *duality map* of Y is a function $J: Y \to Y^*$ satisfying $Jf \in \mathscr{J}f$ for each $f \in Y$.

PROPOSITION. *A necessary and sufficient condition that a linear operator A on Y generates a (C_0) group of isometries on Y is that ± 1 belong to the resolvent set of A and*

$$\operatorname{Re}\langle Af, Jf\rangle = 0$$

for each duality map J of Y and each $f \in \operatorname{Dom}(A)$.

We shall need some information concerning duality maps for Orlicz spaces. A candidate for a duality map of L^Φ is

$$\begin{aligned} &Jf(w) = 0 \quad \text{if} \quad f(w) = 0; \text{ otherwise} \\ &Jf(w) = C_f\|f\|_\Phi^2 \overline{f(w)}\,|f(w)|^{-1}\Phi'(|f(w)|\,\|f\|_\Phi^{-1}) \end{aligned} \tag{1}$$

where $C_f = \left[\int_\Omega |f|\,\Phi'(|f|\,\|f\|_\Phi^{-1})d\mu\right]^{-1}$. Note that J defined by (1) is not a duality map for L^Φ in general; for instance; this is the case if Ω is not a singleton and $L^\Phi = L^\infty$. However, we have the following positive result, which we state for not necessarily atomic measure spaces.

LEMMA. *Let $(\Omega_1, \Sigma_1, \mu_1)$ be an arbitrary measure space with the finite subset property. Suppose that Φ, Ψ are everywhere finite. Let $0 \neq f \in L^\Phi = L^\Phi(\Omega_1, \Sigma_1, \mu_1)$ and define Jf by* (1). *Suppose $Jf \in L^\Psi$ and*

$$\begin{aligned} &\mu_1\{w \in \Omega\colon \Phi \text{ is not differentiable at } |f(w)|\,\|f\|_\Phi^{-1} \text{ or} \\ &\Psi \text{ is not differentiable at } |Jf(w)|\,\|Jf(w)\|_\Psi^{-1}\} = 0\,. \end{aligned} \tag{2}$$

Then $Jf \in \mathscr{J}f$. *If* $\Phi(x) = p^{-1}x^p$, $1 \leqq p < \infty$, $x > 0$, *then* J *defined by* (1) *is a duality map for* L^p.

Proof. This is a variant of a result of Lumer's [9], and the present proof differs from Lumer's. According to general Orlicz space theory (cf. e.g. [17, p. 175]), equality occurs in Hölder's inequality

$$\int_{\Omega_1} f\phi d\mu_1 = ||f||_\Phi ||\phi||_\Psi > 0$$

whenever $\phi = cJf$ for some positive constant c and Jf defined by (1), and in addition,

$$\int_{\Omega_1} \Phi(|f|\,||f||_\Phi^{-1})d\mu_1 = \Phi(1), \qquad \int_{\Omega_1} \Psi(|\phi|\,||\phi||_\Psi^{-1})d\mu_1 = \Psi(1) .$$

These last two conditions hold by (2) together with a slight modification of the proof in [14, p. 682].

Note that (2) automatically holds if both Φ' and Ψ' are continuous. Also, if f is a bounded function in L^Φ which vanishes off a set of finite μ_1-measure, then $Jf \in L^\Psi$.

If $\Phi(x) = p^{-1}x^p$, $1 \leqq p < \infty$, then $L^\Phi = L^p$ and (1) becomes

$$(1') \qquad \begin{aligned} &Jf(w) = 0 \quad \text{if} \quad f(w) = 0; \text{ otherwise} \\ &Jf(w) = ||f||_p^{2-p}\overline{f(w)}\,|f(w)|^{p-2} . \end{aligned}$$

It is easily seen that $Jf \in L^q$ (where $p^{-1} + q^{-1} = 1$) and J defines a duality map for L^p. This follows from the first part of the lemma for $1 < p < \infty$, and from a trivial calculation for $p = 1$; it is also easy to verify this directly.

We shall assume a weak form of the statement: J defined by (1) is a duality map for L^Φ. Specifically, our assumption on Φ is as follows:

(*) (i) $0 < \Phi(x) < \infty$ *for* $x > 0$.

(ii) *If the support of* $f \in L^\Phi$ *consists of at most two points, then* $Jf \in \mathscr{J}f$ *where* Jf *is defined by* (1).

(i) excludes L^∞, but is not otherwise very restrictive. The above lemma gives a sufficient condition for (ii) to hold. In particular, (ii) holds if $L^\Phi = L^p$, $1 \leqq p < \infty$, or if both Φ' and Ψ' are continuous.

3. **The main result.** Let $X = L^\Phi(\Omega, \Sigma, \mu)$ be a real or complex Orlicz space on an atomic measure space, let $T = \{T_t : t \in R\}$ be a (C_0) group of isometries on X, and let A be the infinitesimal generator of T. We shall make the following assumption concerning A.

(**) *For* $w \in \Omega$ *let* $\delta(w)$ *be the function whose value at* $w' \in \Omega$ *is* 1 *or* 0 *according as* $w' = w$ *or* $w' \neq w$. *Assume* $\delta(w) \in \mathrm{Dom}\,(A)$ *for each* $w \in \Omega$.

(**) is automatically satisfied if X is finite dimensional or more generally if T is continuous in the uniform operator topology. Also, (**) is satisfied by all the generators of (C_0) semigroups on the sequence spaces l^{Φ} (or l^p) that one normally encounters in the applications. Therefore, we do not view (**) as being very restrictive.

Our main result is the following

THEOREM. *Let $X = L^{\Phi}(\Omega, \Sigma, \mu)$ be an Orlicz space on an atomic measure space, let $T = \{T_t: t \in \boldsymbol{R}\}$ be a (C_0) group of isometries on X with generator A, and suppose* (*) *and* (**) *hold. Suppose X is not a Hilbert space,* i.e., *Φ is not of the form $\Phi(s) = \text{const} \times s^2$.*

(i) *If X is a real space, then necessarily $T_t = I$ for each $t \in \boldsymbol{R}$.*

(ii) *If X is a complex space, there is a function $g: \Omega \to \boldsymbol{R}$ such that $(T_t f)(w) = \exp\{itg(w)\} f(w)$ for each $f \in X$ and each $w \in \Omega$.*

Note that for any function $g: \Omega \to \boldsymbol{R}$ the formula in (ii) clearly defines a (C_0) group of isometries on X if X is complex.

For the proof of the theorem, suppose that Φ is not of the form $\Phi(s) = \text{const} \times s^2$. We shall prove that for each $w \in \Omega$ there is a real number $g(w)$ such that $A(\delta(w)) = ig(w)\delta(w)$. (In other words, if we view A as a matrix, then all the off diagonal entries are zero, while the diagonal entries are purely imaginary.) The rest of the proof of the theorem runs as follows. Let $f \in X$. Then $f = \sum_{j=1}^{\infty} c_j \delta(w_j)$ for suitable scalars c_j and points $w_j \in \Omega$, since the support of f is at most countable. $f_n = \sum_{j=1}^{n} c_j \delta(w_j) \to f$ as $n \to \infty$ and

$$Af_n = i \sum_{j=1}^{n} c_j g(w_j) \delta(w_j) .$$

Since A is closed it follows that for $f = \sum_{j=1}^{\infty} c_j \delta(w_j) \in \text{Dom}(A)$, $Af = i \sum_{j=1}^{\infty} c_j g(w_j) \delta(w_j)$; and $f = \sum_{j=1}^{\infty} c_j \delta(w_j)$ belongs to Dom (A) if and only if $f \in X$ and there is a $k > 0$ such that $\sum_{j=1}^{\infty} \Phi(k \mid c_j g(w_j) \mid) \mu\{w_j\} < \infty$. The conclusion of the theorem follows immediately.

In order to prove that $A(\delta(w)) = ig(w)\delta(w)$ for some $g(w) \in \boldsymbol{R}$, we assume the contrary and seek a contradiction. First, by the proposition and (*),

$$0 = \text{Re}\, \langle A\delta(w), J\delta(w) \rangle = \text{Re}\, (A(\delta(w))(w)) \Phi(k^{-1}) \mu\{w\}$$

where $k = \| \delta(w) \|_{\Phi}$; whence by (i) of (*), $\text{Re}\,(A(\delta(w))(w)) = 0$, or $A(\delta(w))(w) = ig(w)$ for some $g(w) \in \boldsymbol{R}$. Hence we are assuming that $A(\delta(w_1))(w_2) \neq 0$ for a pair w_1, w_2 of distinct members of Ω, and we seek a contradiction. Let α_1, α_2 be nonzero scalars, let $f_j = \delta(w_j)$, $j = 1, 2$, and let $f = \alpha_1 f_1 + \alpha_2 f_2$. By the proposition, (*) and (**),

$$0 = \mathrm{Re}\,\langle Af, Jf\rangle = \mathrm{Re}\sum_{j=1}^{2}\alpha_j\langle Af_j, Jf\rangle \tag{3}$$
$$= \mathrm{Re}\left\{\sum_{j=1}^{2}\sum_{k=1}^{2}\alpha_j(Af_j)(w_k)\bar{\alpha}_k\,|\alpha_k|^{-1}\,\Phi'(|\alpha_k|\,\|f\|_\Phi^{-1})\mu\{w_k\}\right\}.$$

Letting $\beta_k = |\alpha_k|\,\|f\|_\Phi^{-1}$ and noting that $\mathrm{Re}\,(Af_j)(w_j) = 0$, (3) reduces to

$$0 = \mathrm{Re}\,\{\mu\{w_1\}\Phi'(\beta_1)\alpha_2\bar{\alpha}_1\,|\alpha_1|^{-1}\,(Af_2)(w_1) \tag{4}$$
$$+ \mu\{w_2\}\Phi'(\beta_2)\alpha_1\bar{\alpha}_2\,|\alpha_2|^{-1}\,(Af_1)(w_2)\}.$$

Let $\alpha_2 = z\alpha_1$ with $z \neq 0$. Then (4) becomes

$$0 = \mathrm{Re}\{z[\mu\{w_1\}\Phi'(\beta_1)(Af_2)(w_1) + \mu\{w_2\}\Phi'(|z|\,\beta_1)z^{-2}\,|z|\,(Af_1)(w_2)]\}.$$

Now write $z = re^{i\theta}$ and cancel r from $\mathrm{Re}\,\{\cdots\}$. Recall our assumption that $(Af_1)(w_2) \neq 0$; let θ be the argument of $(Af_1)(w_2)$. Then the second term in $\mathrm{Re}\,\{\cdots\}$ is independent of r, so we conclude that $r^{-1}\Phi'(r\beta_1)\,|(Af_1)(w_2)|$ does not depend on r. Thus $\Phi'(r\beta_1) = cr$ for some real number c and all $r > 0$. Hence $\Phi(s) = (c/2\beta_1)s^2$ for all $s \geqq 0$. This is the desired contradiction, and the proof is complete.

4. Further results and remarks.

COROLLARY. *The conclusions of the theorem hold if X is a finite dimensional L^∞ space* (i.e., *if Ω is a finite set and $X = L^\infty(\Omega, \Sigma, \mu)$*).

Proof. $T^* = \{T_t^*: t \in \boldsymbol{R}\}$ is a (C_0) group of isometries on $X^* = L^1(\Omega, \Sigma, \mu)$. By the theorem, T_t^* is either the identity or multiplication by $\exp\{-itg(\cdot)\}$ for some $g: \Omega \to \boldsymbol{R}$. The proof is completed by taking adjoints again.

REMARK 1. The dual space of $L^\infty(\Omega, \Sigma, \mu)$ can be identified with $L^1(\Omega_1, \Sigma_1, \mu_1)$ for some different measure space (cf. [3, pp. 394–395]). Our proof of the corollary fails to work for the infinite dimensional case for two reasons: $(\Omega_1, \Sigma_1, \mu_1)$ may be nonatomic even if (Ω, Σ, μ) is atomic, and T^* need not be strongly continuous since X is not reflexive.

REMARK 2. Let Σ be a σ-algebra of subsets of $\boldsymbol{R}$ containing all singletons, and let μ be the discrete measure on $(\boldsymbol{R}, \Sigma)$, so that $\mu(E)$ is the number of points in E. Let $X = L^p(\boldsymbol{R}, \Sigma, \mu)$, $1 \leqq p \leqq \infty$. $T = \{T_t: t \in \boldsymbol{R}\}$ defined by $(T_tf)(x) = f(x + t)$ is a group of isometries on X. T is not strongly continuous; it is not even strongly measurable. This follows either from our theorem or from a direct computation. This example shows that the requirement of strong continuity of T in the theorem is essential.

REMARK 3. Lamperti [7] has characterized the isometries and surjective isometries on L^p spaces; in particular, many such exist. Our results show that if the measure space is atomic and if $p \neq 2, \infty$, only the trivial isometries can be embedded in a (C_0) group of isometries. Other aspects of isometries on L^p spaces have been studied recently by Byrne and Sullivan [2]. Lumer [9] has extended Lamperti's results to a large class of Orlicz spaces. Our results similarly complement Lumer's results.

REMARK 4. Following a number of recent authors (cf. for instance [8], [16]), we define a bounded operator A on a Banach space X to be *skew-hermitian* whenever $\| e^{tA} \| = 1$ for each $t \in \boldsymbol{R}$. This is equivalent to saying that the (C_0) group generated by A is a group of isometries. Our theorem thus characterizes the skew-hermitian operators on the Orlicz spaces $L^{\Phi}(\Omega, \Sigma, \mu)$, where the measure space is atomic and Φ satisfies (*). This complements a theorem of Lumer [9, pp. 106–107].

REMARK 5. A (C_0) group of isometries on a finite dimensional L^p space (with $p \neq 2$) is trivial, according to the theorem and corollary above. Nevertheless, there exist nontrivial (C_0) groups of isometries on finite dimensional subspaces of L^p spaces, as the following example shows.

Let $\Omega = \{z \in \boldsymbol{C}: |z| = 1\}$ be the unit circle in the complex plane. Let X be the real space $L^p(\Omega, \Sigma, \mu)$, where Σ is the σ-algebra of Borel sets of Ω, μ is Haar measure, and $1 \leqq p < \infty$. $T = \{T_t: t \in \boldsymbol{R}\}$ is a (C_0) group of isometries on X, where

$$(T_t f)(x) = f(x + t)$$

for $f \in X$, $t, x \in \boldsymbol{R}$; here we are regarding members of X as (equivalence classes of) 2π-periodic real functions on $\boldsymbol{R}$. Let $Y = \{f_a: a = (a_1, a_2) \in \boldsymbol{R}^2\}$ where for $a \in \boldsymbol{R}^2$, $x \in \boldsymbol{R}$, $f_a(x) = a_1 \cos x + a_2 \sin x$. Y is a two dimensional subspace of X left invariant by T_t for each real t (as a simple computation shows). Thus the restriction of T to Y is the desired example.

We note, incidentally, that the closed subspaces of L^p which are themselves L^p spaces have been (almost completely) characterized by Ando [1] and Tzafriri [15].

ACKNOWLEDGEMENT. The example in Remark 5 is due to Marek Kanter. I wish to acknowledge with pleasure a number of discussions with Kanter which provided the motivation for the present work. Kanter has obtained results which complement those of the present paper (including a characterization of the (C_0) groups of

isometries on finite dimentional subspaces of L^p spaces), and his methods are probabilistic. Also, after this paper was submitted for publication, Kanter discovered Lumer's paper [9] and informed me of its existence. Finally, I thank M. M. Rao for some helpful comments.

REFERENCES

1. T. Ando, *Contractive projections in L_p spaces*, Pacific J. Math., **17** (1966), 391-405.
2. C. Byrne and F. E. Sullivan, *Contractive projections with contractive complements in L_p space*, J. Multivariate Analysis, to appear.
3. N. Dunford and J. T. Schwartz, *Linear Operaters, Part I: General Theory*, Interscience, New York, 1958.
4, J. A. Goldstein, *Semigroups of Operators and Abstract Cauchy Problems*, Tulane University Lecture Notes, New Orleans, 1970.
5. E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups of Operators*, Amer. Math. Soc., Providence, R. I., 1957.
6. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Gronigen, 1961.
7. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math., **8** (1958), 459-466.
8. G. Lumer, *Semi-inner product spaces*, Trans. Amer. Math. Soc., **100** (1961), 29-43.
9. ———, *On the isometries of reflexive Orlicz spaces*, Ann. Inst. Fourier, Grenoble, **13** (1963), 99-109.
10. G. Lumer and R. S. Phillips, *Dissipative operators on a Banach space*, Pacific J. Math., **11** (1961), 679-698.
11. W. A. J. Luxemburg, *Banach Function Spaces*, Thesis, Delft, 1955.
12. M. M. Rao, *Linear functionals on Orlicz spaces*, Nieuw Arch. Wisk., **12** (1964), 77-98.
13. ———, *Linear functionals on Orlicz spaces: general theory*, Pacific J. Math., **25** (1968), 553-585.
14. ———, *Smoothness of Orlicz spaces*, Indag. Math., **68** (1965), 671-690.
15. L. Tzafriri, *Remarks on contractive projections in L_p-spaces*, Israel J. Math., **7** (1969), 9-15.
16. I. Vidav, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Z., **66** (1956), 121-128.
17. A. Zygmund, *Trigonometrical Series*, Vol. I (2nd ed.), Cambridge University Press, 1959.

Received December 14, 1971 and in revised form January 6, 1972. Supported by NSF grant GP-28652.

TULANE UNIVERSITY