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SOME COMMUTANTS IN $B(c)$ WHICH ARE ALMOST MATRICES

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We determine necessary and sufficient conditions for two linear operators in $B(c)$ to commute. Specializing one of the operators to be a conservative triangular matrix we determine that most such operators have commutants consisting of almost matrices of a special form.

Almost matrices were developed in [10] for reasons not related to this paper, but they find application here in that the commutants in $B(c)$ of certain matrices must be almost matrices.

Let c denote the space of convergent sequences, $B(c)$ the algebra of all bounded linear operators over c, e the sequence of all ones, and e^k the coordinate sequences with a one in the kth position and zeros elsewhere. If $T \in B(c)$, then one can define continuous linear functionals χ and χ_i by $\chi(T) = \lim Te - \sum_k \lim (Te^k)$ and $\chi_i(T) = (Te)_i - \sum_k (Te^k)_i$, $i = 1, 2, \cdots$. (See, e.g. [9, p. 241].) It is known [1, p. 8] that any $T \in B(c)$ has the representation $T = v \otimes \lim + B$, where B is the matrix representation of the restriction of T to c_0, the subspace of null sequences, v is the bounded sequence $v = \{\chi_i(T)\}$, and $v \otimes \lim x = (\lim x)v$ for each $x \in c$.

The second adjoint of T (see, e.g. [1, p. 8] or [10, p. 357]) has the matrix representation

$$(*) \qquad T'' = \begin{pmatrix} \chi(T) & b_1 & b_2 & \cdot & \cdot & \cdot \\ \chi_1(T) & b_{11} & b_{12} & \cdot & \cdot & \cdot \\ \chi_2(T) & b_{21} & b_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where the b_i's occur in the representation of $\lim \circ T \in c'$ as $(\lim \circ T)(x) = \lim (Tx) = (T) \lim x + \sum_k b_k x_k$; namely, $b_i = \lim Te^i$. With the use of $(*)$ it is easy to describe the commutant of any $Q \in B(c)$.

THEOREM 1. *Let $Q = u \otimes \lim + A \in B(c)$. Then* Com (Q) *in* $B(c) = \{T = v \otimes \lim + B \in B(c)$: *$T$ satisfies* (1)–(3)$\}$, *where*

$$(1) \qquad u_n \chi(T) + \sum_{k=1}^{\infty} a_{nk} v_k = v_n \chi(Q) + \sum_{k=1}^{\infty} b_{nk} u_k \, ; \qquad n = 1, 2, \cdots$$

$$(2) \qquad u_n b_k + \sum_{j=1}^{\infty} a_{nj} b_{jk} = v_n a_k + \sum_{j=1}^{\infty} b_{nj} a_{jk} \, ; \qquad n, k = 1, 2, \cdots$$

$$(3) \qquad \sum_{k=1}^{\infty} a_k v_k = \sum_{k=1}^{\infty} b_k u_k \ ,$$

and where $a_k = \lim Q(e^k)$, $b_k = \lim T(e^k)$.

To prove Theorem 1, use the representation $(*)$ for T'' and Q'' and then equate the corresponding terms in the products $T''Q''$ and $Q''T''$. For example, (1) is obtained by equating $(Q''T'')_{n1}$ and $(T''Q'')_{n1}$. When Q is a matrix A, each $u_n = 0$ and each $a_k = \lim_n a_{nk}$. The following result is an immediate consequence of Theorem 1.

COROLLARY 1. *Let A be a conservative matrix, $T \in B(c)$. Then $A \leftrightarrow T$ if and only if*

$$(4) \qquad Av = \chi(A)v$$

$$(5) \qquad \sum_{j=1}^{\infty} a_{nj} b_{jk} = v_n a_k + \sum_{j=1}^{\infty} b_{nj} a_{jk} \ ; \qquad n, k = 1, 2, \cdots$$

$$(6) \qquad a \perp v, \text{ where } a = \{a_n\} \ .$$

A conservative matrix A is called multiplicative if $\lim_A x = \chi(A) \lim x$ for each $x \in c$; i.e., if each $a_k = 0$.

COROLLARY 2. *Let A be a conservative multiplicative matrix. Then $A \leftrightarrow T$ if and only if A satisfies* (4) *and*

$$(7) \qquad B \longleftrightarrow A \ .$$

If A is multiplicative, then each $a_k = 0$ and condition (5) of Corollary 1 reduces to (7) of Corollary 2. Since $a = 0$, (6) holds automatically.

THEOREM 2. *Let A be a conservative matrix. Then $A \leftrightarrow v \otimes$* lim *if and only if*

$$(8) \qquad (\lim x)Av = (\lim_A x)v \text{ for each } x \in c \ .$$

To establish (8) note that $A(v \otimes \lim)(x) = A(\lim x)v = (\lim x)Av$, and $(v \otimes \lim)(Ax) = (\lim Ax)v = (\lim_A x)v$.

COROLLARY 3. *Let A be a conservative multiplicative matrix. Then $A \leftrightarrow u \otimes$* lim *if and only if A satisfies* (4).

COROLLARY 4. *Let A be a conservative multiplicative matrix. Then $A \leftrightarrow T$ if and only if $A \leftrightarrow v \otimes$* lim *and $A \leftrightarrow B$.*

For $T \in B(c)$, T is called an almost matrix if $v \in c$. A matrix A

is called triangular if $a_{nk} = 0$ for each $k > n$. We shall now examine some triangular matrices whose commutants consist of almost matrices.

THEOREM 3. *Let A be a conservative triangular matrix with $a_{nn} \neq \chi(A)$ for $n > 1$. Consider the conditions*

$$\sum_{k=1}^{n} a_{nk} = \chi(A) \text{ for } n > 1 \tag{9}$$

$$T \leftrightarrow A \text{ implies } T \text{ is an almost matrix with } v = \lambda e\,. \tag{10}$$

Then (9) $\Rightarrow$ (10). *If, in addition,* $\lambda \neq 0$, *then* (10) $\Rightarrow$ (9).

To prove that (9) $\Rightarrow$ (10), suppose $T \leftrightarrow A$. From (4) of Corollary 1,

$$\sum_{k=1}^{n} a_{nk} v_k = \chi(A) v_n = \Big(\sum_{k=1}^{n} a_{nk}\Big) v_n\,, \qquad n > 1\,.$$

We may rewrite the equation in the form $\sum_{k=1}^{n} (v_k - v_n) a_{nk} = 0$, which, along with the hypothesis $a_{nn} \neq \chi(A)$ for $n > 1$, yields $v_n = v_1$, for $n > 1$.

For $n > 1$, $(T''A'')_{n+1,1} = \lambda\chi(A)$ and $(A''T'')_{n+1,1} = \lambda \sum_{k=1}^{n} a_{nk}$. Thus, if $\lambda \neq 0$, $\chi(A) = \sum_{k=1}^{n} a_{nk}$.

The result stated at the end of paragraph 2 in the next section shows that the condition $\lambda \neq 0$ is necessary for (10) to imply (9).

The identity matrix shows that the restriction $a_{nn} \neq \chi(A)$ for $n > 1$ cannot be removed.

COROLLARY 5. *Let A be a conservative triangular matrix with $\sum_{k=1}^{n} a_{nk} = \chi(A)$ for $n > 1$ and $a_{nn} \neq \chi(A)$ for each n. Then $T \leftrightarrow A$ implies T is a matrix.*

From Theorem 3, $v_n = v_1$. From (4) with $n = 1$ we get $a_{11}v_1 = \chi(A)v_1$. Since $a_{11} \neq \chi(A)$, $v_1 = 0$ and A is a matrix.

Applications. 1. Let C denote the Casàro matrix of order 1. Then Theorem 3 of [7] follows immediately from Theorem 3 of this paper.

2. Endl [2], Hausdorff [4], Jakimovski [5] (see [11, p. 190]) and Leininger [6] have defined summability methods which are generalizations of the Hausdorff methods. The $(H, \lambda_n; \mu_n)$ transform of [5] is defined by a triangular matrix $H = (h_{nk})$ with entries $h_{nn} = \mu_n$, $h_{nk} = (-1)^{n-k}\lambda_{k+1} \cdots \lambda_n[\mu_k, \cdots, \mu_n]$, $k < n$, where

$$[\mu_k, \cdots, \mu_n] = \sum_{i=k}^{n} \frac{\mu_i}{(\lambda_i - \lambda_k) \cdots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \cdots (\lambda_i - \lambda_n)}\,,$$

$\{\mu_n\}$ is a real or complex sequence, and $\{\lambda_n\}$ satisfies $0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, $\lim_n \lambda_n = \infty$ and $\sum_i \lambda_i^{-1} = \infty$. If $\lambda_n = n$, $n \geqq 0$, then $(H, \lambda_n; \mu_n)$ reduces to the ordinary Hausdorff transformations.

[4] is a special case of [5] with $\lambda_0 = 0$. [2] is the special case of [5] with $\lambda_n = n + \alpha$.

Each conservative method $(H, \lambda_n; \mu_n)$ with distinct diagonal entries and $\lambda_0 = 0$ satisfies the conditions of Theorem 3. Thus, if $T \leftrightarrow (H, \lambda_n; \mu_n)$; T is an almost matrix with $v = \lambda e$. If, in addition, $(H, \lambda_n; \mu_n)$ satisfies condition (1) of [7], then $T \leftrightarrow (H, \lambda_n; \mu_n)$ implies that B is a generalized Hausdorff matrix of the same type as $(H, \lambda_n; \mu_n)$.

If $\lambda_0 > 0$, then (9) of Theorem 3 is not satisfied. However, $\lim_n \sum_k h_{nk} = \mu_0$, and one can establish the following: Let $(H, \lambda_n; \mu_n)$ be a multiplicative generalized Hausdorff matrix with $\lambda_0 > 0$ and $\mu_n \neq \mu_0$ for all $n > 0$. Then Com $(H, \lambda_n; \mu_n)$ in Γ = Com $(H, \lambda_n; \mu_n)$ in $B(c)$.

The commutant question for the matrices of [6] remains open.

3. Let A be the shift, i.e., $a_{n+1,n} = 1$, $a_{nk} = 0$ otherwise. Then Theorem 1.1 of [8] follows from Corollary 5.

4. Let A be any regular Nörlund method with $p_n > 0$ for all n. (A matrix A is said to be regular if $\lim_A x = \lim x$ for each $x \in c$.) Then, by Theorem 3, if $T \leftrightarrow A$ then T is an almost matrix with $v = \lambda e$.

5. A triangle is a triangular matrix with each $a_{nn} \neq 0$. A factorable triangular matrix has entries of the form $a_{nk} = c_k d_n$, $k \leqq n$. Let A be a regular factorable triangle with all row sums one. By Theorem 3, if $T \leftrightarrow A$, then T is an almost matrix with $v = \lambda e$. This result holds, in particular, for the weighted mean methods (see [3, p. 57]).

THEOREM 4. *Let A be a conservative triangular matrix with $\sum_{k=1}^n a_{nk} = \chi(A)$ for each n, and $a_{nn} \neq \chi(A)$ for $n > 1$. Then the following are equivalent*:

(i) *A is multiplicative.*

(ii) *$T \leftrightarrow A$ if and only if there exists a scalar $\lambda \neq 0$ such that $T = \lambda e \otimes \lim + B$, where $B \leftrightarrow A$.*

(i) $\Rightarrow$ (ii). Suppose $T \leftrightarrow A$. By Corollary 2 we have (4) and $B \leftrightarrow A$. The hypotheses then allow us to use Theorem 3. Suppose now that T has the indicated form. Since $v = \lambda e$ and $\sum_{k=1}^n a_{nk} = \chi(A)$ for each n, A satisfies (4). By Corollary 2, $A \leftrightarrow T$.

(ii) $\Rightarrow$ (i). Using Corollary 4 and Theorem 2 we have (8). Set $x = e^k$ to get $a_k = 0$ for each k, since $\lambda \neq 0$. Thus A is multiplicative.

Note that the condition $\lambda \neq 0$ is not used in the proof of (i) $\Rightarrow$ (ii). However, it is necessary for the converse. For, let H denote the Hausdorff matrix generated by $\mu_n = n(n+1)^{-1}$, K the compact Hausdorff matrix generated by $\{1, 0, 0, \cdots\}$. Then, since $H = I - C$; where C is the Cesàro matrix of order 1, $A \leftrightarrow H$ if and only if $A \leftrightarrow C$. But $K \leftrightarrow C$. Therefore, $K \leftrightarrow H$ and K is not multiplicative.

The condition $\sum_{k=1}^{n} a_{nk} = \chi(A)$ for each n cannot be removed. For example, let A be the matrix defined by $a_{11} = 1$, $a_{2n+1,2n-1} = 1$, $a_{2n,2n} = (n+1)/n$, $n = 1, 2, \cdots, a_{nk} = 0$ otherwise. Let T be the operator with $v_{2n-1} = 1$, $v_{2n} = 0$, and B a diagonal matrix with $b_{2n,2n} = 1$, $b_{2n-1,2n-1} = 0$. Then $T \in B(c)$, A is regular, $a_{nn} \neq 1 = \chi(A)$ for any n, and $A \leftrightarrow T$, but T is not an almost matrix.

COROLLARY 6. *Let A satisfy the hypotheses of Theorem 4 with $\chi(A) = 1$. Then the following are equivalent:*

(i) *A is regular.*

(ii) *$T \leftrightarrow A$ if and only if there exists a scalar $\lambda \neq 0$ such that $T = \lambda e \otimes \lim + B$, where $B \leftrightarrow A$.*

In Theorem 4 merely observe that the conditions A multiplicative and $\chi(A) = 1$ imply A is regular.

A natural question to ask is whether there exist matrices whose commutant in $B(c)$ not only contains almost matrices different from those with $v = \lambda e$, but also such that Com(A) in $B(c)$ is included in the set of almost matrices. The answer is yes, as the following example illustrates.

Let v be a positive nonconstant convergent sequence with $v_n \neq 0$ for any n, $\lim_n v_n \neq 0$, $v_n/v_{n-1} \leqq 1$ for all n, and $\lim_n v_{n+1}/v_n = 1$. Let A be the matrix defined by $a_{11} = 1$, $a_{n,n-1} = v_n/v_{n-1}$, $n > 1$, $a_{nk} = 0$ otherwise. We wish to show that $A \leftrightarrow T = v \otimes \lim + B$, where $B \leftrightarrow A$. From Corollary 2 we need to verify (4) and (7).

To verify (4) for $n = 1$, $a_{11}v_1 = v_1 = \chi(A)v_1$. For $n > 1$, $A_n(v) = a_{n,n-1}v_{n-1} = v_n = \chi(A)v_n$.

It remains to determine those matrices B which commute with A. It is not difficult, using the techniques of [7], to show that Com(A) in $\Delta =$ Com(A) in Γ.

We shall now show that Com$(A) = \{f(A): f$ is analytic in $D = \{z: |z| \leqq 1\}\}$.

For convenience set $\alpha_n = v_{n+1}/v_n$. Suppose $B \leftrightarrow A$. Equating $(BA)_{n,k-1}$ and $(AB)_{n,k-1}$ we get, for $k > 2$,

$$b_{nk} = \frac{\alpha_{n-1}\alpha_{n-2} \cdots \alpha_{n-k+2}}{\alpha_{k-1} \cdots \alpha_2} b_{n-k+2,2} .$$

Thus we may write

$$(11) \qquad b_{n,n-k} = \alpha_{n-1}\alpha_{n-2} \cdots \alpha_{n-k}\lambda_k , \qquad 1 \leqq k \leqq n-2 ,$$

where $\lambda_k = b_{k+2,2}/\alpha_{k+1} \cdots \alpha_2,\ k \geqq 1$.

For $r = 1, 2, \cdots$,

$$(A^r)_{n,n-k} = \begin{cases} 1 & , \quad n-k=1,\ k=1 \\ \alpha_1\alpha_2 \cdots \alpha_{n-1} & , \quad n-k=1<n \leqq r+1 \\ \alpha_{n-1} \cdots \alpha_{n-r} & , \quad r=k \\ 0 & , \quad \text{otherwise}. \end{cases}$$

Note that for $n-k>1$, the only nonzero entries of A^r occur on the rth diagonal. Thus for any n, there exists only one nonzero element in row n. With λ_0 any arbitrary scalar, and for any fixed n, k with $n-k>1$, $\sum_{j=0}^{\infty} \lambda_j(A^j)_{n,n-k}$ has at most two nonzero terms. One is $\lambda_k(A^k)_{n,n-k}$ and the other is $\lambda_0\delta_{n-k}^{n}$. Therefore,

$$\sum_{j=0}^{\infty} \lambda_j(A^j)_{n,n-k} = \left(\sum_{j=0}^{\infty} \lambda_j A^j\right)_{n,n-k} = (f(A))_{n,n-k}\ .$$

For $n-k=1,\ n>1$,

$$\sum_{j=0}^{\infty} \lambda_j(A^j)_{n1} = \sum_{j=n}^{\infty} \lambda_j(\alpha_1\alpha_2 \cdots \alpha_{n-1}) = (f(A))_{n1}\ .$$

For $n-k=1,\ n=1$,

$$\sum_{j=0}^{\infty} \lambda_j(A^j)_{11} = \sum_{j=0}^{\infty} \lambda_j = (f(A))_{11}\ ,$$

assuming $\sum_j \lambda_j$ converges, so that $B = f(A)$.

Using (11), we may write $\lambda_k = b_{n,n-k}/\alpha_{n-1}\alpha_{n-2} \cdots \alpha_{n-k}$; since $\alpha_1 \cdots \alpha_n = u_{n+1}/u_1$, we have

$$\sum_{k=1}^{n} |\lambda_k| = \sum_{k=1}^{n} \left| \frac{u_{n-k}}{u_n} b_{n,n-k} \right| = \frac{1}{u_n} \sum_{k=1}^{n} u_k |b_{nk}|\ .$$

Since $\|B\| < \infty$ and $\{u_n\}$ is bounded away from zero, $f(z) = \sum_j \lambda_j z^j$ is analytic in D.

Conversely, if B has the form $f(A)$ for some f analytic in D, then clearly B commutes with A.

We conclude with a few remarks concerning conull matrices. A conservative matrix is conull if $\chi(A) = 0$. From (4) of Corollary 1, $Av = 0$. Therefore, Com (A) in $B(c) = \{T \in B(c): v \in \text{null space of } A\}$. If A is a triangle, then $v = 0$ and Com (A) in $B(c) =$ Com (A) in Γ. If A is triangular, with only a finite number of zeros on the main diagonal, then $v \in$ linear span $(e_1, e_2, \cdots, e_n)$, where n is the largest integer for which $a_{nn} = 0$. Of course, if A is the zero matrix, then Com (A) in $B(c) = B(c)$.

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