PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

# MAXIMAL INVARIANT SUBSPACES OF STRICTLY CYCLIC OPERATOR ALGEBRAS

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**A *strictly cyclic* operator algebra $\mathscr{A}$ on a complex Banach space $X(\dim X \geqq 2)$ is a uniformly closed subalgebra of $\mathscr{L}(X)$ such that $\mathscr{A}x = X$ for some $x$ in $X$. In this paper it is shown that (i) if $\mathscr{A}$ is strictly cyclic and intransitive, then $\mathscr{A}$ has a maximal (proper, closed) invariant subspace and (ii) if $A \in \mathscr{L}(X)$, $A \neq zI$ and $\{A\}'$ (the commutant of $A$) is strictly cyclic, then $A$ has a maximal hyperinvariant subspace.**

1. **Notation and terminology.** Throughout the paper $X$ is a complex Banach space of dimension greater than one and $\mathscr{L}(X)$ is the algebra of continuous linear operators on $X$. $\mathscr{A}$ will denote a uniformly closed subalgebra of $\mathscr{L}(X)$ which is *strictly cyclic* and $x_0$ will be a *strictly cyclic vector* for $\mathscr{A}$: that is, $\mathscr{A}x_0 = X$. We do not insist that the identity element $I$ of $\mathscr{L}(X)$ be an element of $\mathscr{A}$.

If $\mathscr{B} \subset \mathscr{L}(X)$, then the *commutant* of $\mathscr{B}$ is $\mathscr{B}' = \{E: E \in \mathscr{L}(X)$ and $EB = BE$ for all $B$ in $\mathscr{B}\}$. We shall use the terminology of "invariant" and "transitive" as follows: if $M \subset X$ and $\mathscr{B} \subset \mathscr{L}(X)$, then (i) $M$ is *invariant* under $\mathscr{B}$ if $\mathscr{B}M = \{Bm: B \in \mathscr{B}$ and $m \in M\} \subset M$, (ii) $M$ is an *invariant subspace* for $\mathscr{B}$ if $M$ is invariant under $\mathscr{B}$ and $M$ is a closed, nontrivial ($\neq \{0\}$, $X$) linear subspace of $X$, (iii) $\mathscr{B}$ is *transitive* if $\mathscr{B}$ has no invariant subspace and *intransitive* if $\mathscr{B}$ has an invariant subspace. Further, if $A \in \mathscr{L}(X)$ and $\{A\}'$ is intransitive, then each invariant subspace of $\{A\}'$ is called a *hyperinvariant subspace* of $A$. Finally an invariant subspace of $\mathscr{B}$ is *maximal* if it is not properly contained in another invariant subspace of $\mathscr{B}$.

2. **Introduction.** Strictly cyclic operator algebras have been studied by A. Lambert, D. A. Herrero, and the auther of this paper. (See for example [2]-[6].) One of the major results in [2, Theorem 3.8], [3, Theorem 2], and [6, Theorem 4.5] is that a transitive subalgebra of $\mathscr{L}(X)$ containing a strictly cyclic algebra is necessarily strongly dense in $\mathscr{L}(X)$. In each of three developments the following is a key lemma: The only dense linear manifold invariant under a strictly cyclic subalgebra of $\mathscr{L}(X)$ is $X$. In Lemma 1 we shall present a generalization of this lemma which will be useful in the study of maximal invariant subspaces and noncyclic vectors of a strictly cyclic algebra $\mathscr{A}$.

LEMMA 1. *If $M$ is invariant under $\mathscr{A}$ and $x_0 \in \bar{M}$, then $M = X$.*

(It should be noted that we do not require $M$ to be linear nor do we require, as was done in Lemma 3.4 of [2], that $I \in \mathscr{A}$. The proof given here is a slight modification of that given in [2].)

*Proof.* We shall show that $\mathscr{A}x_0 \subset M$ and thus $X = \mathscr{A}x_0 \subset M$. Let $\{x_n\}$ be a sequence in $M$ such that $\lim_{n\to\infty} x_n = x_0$. By [2, Lemma 3.1 (ii)] there exists a sequence $\{A_n\}$ in $\mathscr{A}$ such that $A_n x_0 = x_0 - x_n$ and $\lim_{n\to\infty} \|A_n\| = 0$. Thus for $n$ sufficiently large, $\|A_n\| < 1$ and $(I - A_n)^{-1} = \sum_{k=0}^{\infty} (A_n)^k$. Consequently, $\mathscr{A}(I - A_n)^{-1} \subset \mathscr{A}$ and since $x_0 = (I - A_n)^{-1}x_n$, we have $\mathscr{A}x_0 = \mathscr{A}(I - A_n)^{-1}x_n \subset \mathscr{A}x_n \subset M$, as desired.

For the sake of future reference we restate and reprove the transitivity theorem.

THEOREM 1. *If $\mathscr{A}$ is a strictly cyclic transitive subalgebra of $\mathscr{L}(X)$, then $\mathscr{A}$ is strongly dense in $\mathscr{L}(X)$.*

*Proof.* Using Lemma 1 we can show (as in [2, Lemma 3.5]) that each densely defined linear transformation commuting with $\mathscr{A}$ is everywhere defined and continuous. Further, again using Lemma 1, we can show that if $E \in \mathscr{A}$ and $z \in \sigma(E)$, then either $zI - E$ is not one-to-one or does not have dense range. Thus if $\mathscr{A}$ is transitive, necessarily $E = zI$. Consequently, it follows from [1, p. 636 and Cor. 2.5, p. 641] that $\mathscr{A}$ is strongly dense in $\mathscr{L}(X.)$

3. Maximal invariant subspaces. In [2, Theorem 3.1] it is shown that every strictly cyclic, separated operator algebra $\mathscr{A}$ has a maximal invariant subspace. ($\mathscr{A}$ is separated by $x_0$ if $A = 0$ whenever $A \in \mathscr{A}$ and $Ax_0 = 0$.) Theorem 2 allows us to obtain the same result without the hypothesis that $\mathscr{A}$ be separated, provided $\mathscr{A}$ is intransitive.

THEOREM 2. *An intransitive, strictly cyclic subalgebra $\mathscr{A}$ of $\mathscr{L}(X)$ has a maximal invariant subspace.*

*Proof.* Let $\mathscr{M} = \{M: M$ is an invariant subspace of $\mathscr{A}\}$. By hypothesis $\mathscr{M} \neq \emptyset$. We shall order $\mathscr{M}$ by set inclusion and show that each linearly ordered subset of $\mathscr{M}$ has an upper bound in $\mathscr{M}$. To this end we let $\{M_\alpha\}$ be a linearly ordered subset of $\mathscr{M}$. Then $\bigcup_\alpha M_\alpha$ is invariant under $\mathscr{A}$. By Lemma 1, if $\overline{\bigcup_\alpha M_\alpha} = X$, then $\bigcup_\alpha M_\alpha = X$ and consequently $x_0 \in M_\alpha$ for some value of $\alpha$. Since this last implies that $X = \mathscr{A}x_0 \subset \mathscr{A}M_\alpha \subset M_\alpha$ and contradicts the fact that $M_\alpha$ is a proper closed linear subspace of $X$, we see that $\bigcup_\alpha M_\alpha$ is not dense in $X$. Thus $\overline{\bigcup_\alpha M_\alpha}$ is an element of $\mathscr{M}$ and is an upper bound for $\{M_\alpha\}$. By the Maximality Principle $\mathscr{M}$ has a maximal element.

Lemma 1 and the Maximality Principle can be combined to arrive at other similar results. For example, (i) if $\mathscr{A}$ is intransitive and strictly cyclic, then $\mathscr{A}$ has a proper maximal invariant subset (this will be discussed further in §4) and (ii) if $X$ is a Hilbert space and $\mathscr{A}$ has a reducing subspace (that is, an invariant subspace of $\mathscr{A}$ which is also invariant under $\mathscr{A}^* = \{A^* : A \in \mathscr{A}\}$), then $\mathscr{A}$ has a maximal reducing subspace.

In [2, Theorem 3.7] it is shown that if $A$ is not a scalar multiple of $I$ and $\{A\}'$ is strictly cyclic, then $A$ has a hyperinvariant subspace. This result combined with Theorem 2 yields the following:

COROLLARY 1. *If $A$ is not a scalar multiple of $I$ and $\{A\}'$ is strictly cyclic, then $A$ has a maximal hyperinvariant subspace.*

We shall now turn our attention to intransitive, strictly cyclic operator algebras on a Hilbert space $X$. If $M$ is a closed linear subspace of $X$, $P_M$ will denote the orthogonal projection of $X$ onto $M$ and $M^\perp$ the orthogonal complement of $M$: $M^\perp = \{y : \langle y, m\rangle = 0$ for all $m$ in $M\}$. Furthermore, $\mathscr{A}^* = \{A^* : A \in \mathscr{A}\}$.

In the Hilbert space situation we are able to conclude that $\mathscr{A}^*/M$ is strongly dense in $\mathscr{L}(M^\perp)$ when $M$ is a maximal invariant subspace for $\mathscr{A}$. This remains an open question if $X$ is an arbitrary Banach space and is a particularly interesting one if $X$ is reflexive. For in that case if $M$ is a maximal invariant subspace of $\mathscr{A}$, then $M^\perp = \{x^* : x^*(M) = 0\}$ is a minimal invariant subspace of $\mathscr{A}^*$.

THEOREM 3. *Let $\mathscr{A}$ be a strictly cyclic operator algebra on a Hilbert space $X$. If $M$ is a maximal invariant subspace of $\mathscr{A}$, then*
(i) $(I - P_M)\mathscr{A}(I - P_M)x_0 = M^\perp$ *and* (ii) $\mathscr{A}^*(I - P_M)$ *is strongly dense in $\mathscr{L}(M^\perp)$.*

*Proof.* Note first that $(I - P_M)\mathscr{A}(I - P_M) = (I - P_M)\mathscr{A}$, so that (i) is immediate. Since $M$ is a maximal invariant subspace for $\mathscr{A}$, $M^\perp$ is a minimal invariant subspace for $\mathscr{A}^*$. Thus each of $\mathscr{A}^*(I - P_M)$ and $(I - P_M)\mathscr{A}(I - P_M)$ is transitive on $M^\perp$. Thus the uniform closure of $(I - P_M)\mathscr{A}(I - P_M)$ in $\mathscr{L}(M^\perp)$ is transitive and by (i) is strictly cyclic; hence by Theorem 1 $(I - P_M)\mathscr{A}(I - P_M)$ is strongly dense in $\mathscr{L}(M)$, which concludes our proof of (ii).

THEOREM 4. *Let $X$ be a Hilbert space, $A \in \mathscr{L}(X)$ and $\{A\}'$ strictly cyclic. If $M$ is a maximal invariant subspace for $\{A\}'$, then there exists a multiplicative linear functional $f$ on $\{A\}''$ such* *that for each $E$ in $\{A\}''$, $(E - f(E)I)(X) \subset M$.*

*Proof.* As we noted in the proof of Theorem 3,

$$\mathscr{B} = (I - P_M)\{A\}'(I - P_M)$$

is strongly dense in $\mathscr{L}(M^\perp)$ and thus its commutant consists of the scalar multiples of the identity operator on $M^\perp$. Since $\{A\}'' \subset \{A\}'$ and $M$ is invariant under $\{A\}'$, we know that $(I - P_M)\{A\}''(I - P_M)$ is contained in the commutant of $\mathscr{B}$ on $M^\perp$ and hence $(I - P_M)\{A\}''(I - P_M) \subset \{z(I - P_M)\}$. Thus for $E$ in $\{A\}''$, there exists a complex number $z$ such that $(I - P_M)E(I - P_M) = z(I - P_M)$. Therefore, $(I - P_M)(E - zI) = 0$ since $M$ is invariant under $\{A\}''$; or equivalently $(E - zI)(X) \subset M$. Since $M$ is a proper subset of $X$, it is now obvious that the number $z$ for which $(E - zI)(X) \subset M$ is unique. Define $f(E) = z$.

That $f$ is linear follows immediately from the fact that $f(E)$ is the unique number for which $(E - f(E)I)(X) \subset M$. Furthermore, since $M$ is invariant under $\{A\}''$, $(FE - f(E)F)(X) \subset M$ for all $E, F \in \{A\}''$. Consequently (by uniqueness again), $0 = f(FE - f(E)F) = f(FE) - f(E)f(F)$ and thus we see that $f$ is multiplicative.

COROLLARY 2. *Let $A \in \mathscr{L}(X)$ where $X$ is a Hilbert space. If the range of $A - zI$ is dense in $X$ for each complex $z$, then $\{A\}'$ is not strictly cyclic.*

*Proof.* Except for one minor technicality, Corollary 2 follows immediately from Theorem 4. For, if $\{A\}'$ is strictly cyclic and intransitive, by Theorem 4 there exists a complex number $f(A)$ such that the range of $A - f(A)I$ is contained in a proper subspace of $X$. By Corollary 1 the only other way in which $\{A\}'$ can be strictly cyclic is when $A = zI$ for some complex $z$, in which case the range of $A - zI$ is certainly not dense in $X$.

In [2, Lemma 3.6] and [3, Proposition 2], it is shown that if $E \in \mathscr{A}'$, where $\mathscr{A}$ is strictly cyclic and $z \in \sigma(E)$, then either $zI - E$ is not one-to-one or $zI - E$ does not have dense range. Corollary 2 now adds to our knowledge of $\sigma(A)$ where $\{A\}'$ is strictly cyclic: in this case we know that for at least one value of $z$, the range of $A - zI$ is nondense. Indeed we have the stronger result:

COROLLARY 3. *Let $A \in \mathscr{L}(X)$ where $X$ is a Hilbert space. If $\{A\}'$ is strictly cyclic, then there exists a common eigenvector for $\{A^*\}''$.*

*Proof.* The case in which $\{A\}' = \mathscr{L}(X)$ is trivial. Thus we assume $A \neq zI$. By Theorem 4 if $E \in \{A\}''$, there exists a complex number $f(E)$ such that $(E - f(E)I)(X) \subset M$ where $M$ is a maximal invariant subspace of $\{A\}'$. Therefore, $E^*(I - P_M)x_0 = f(E)^*(I - P_M)x_0$ and $(I - P_M)x_0 \neq 0$ since $x_0$ is cyclic for $\{A\}'$ and $M$ is a proper invariant subspace for $\{A\}'$.

**4. Noncyclic vectors of $\mathscr{A}$.** In this last section of this paper we shall discuss briefly several properties of the set of noncyclic vectors of a strictly cyclic operator algebra $\mathscr{A}$. A vector $x$ is noncyclic for $\mathscr{A}$ if $\mathscr{A}x$ is not dense in $X$. These results are summarized in Theorem 5. Parts (i) and (iii) of Theorem 5 also are found in [5, Theorem 2].

THEOREM 5. *Let $N$ be the set of noncyclic vectors of a strictly cyclic operator algebra $\mathscr{A}$,*

(i) *if $x \notin N$, then $x$ is a strictly cyclic vector for $\mathscr{A}$,*

(ii) *$N$ is invariant under $\mathscr{A}$,*

(iii) *$N$ is closed in $X$,*

(iv) *$N$ is the unique proper maximal invariant subset of $\mathscr{A}$,*

(v) *if $N$ is not linear, then $N + N = X$, where $N + N = \{x + y: x, y \in N\}$.*

*Proof.* (i) If $x \notin N$, then $\overline{\mathscr{A}x} = X$ and thus by Lemma 1 since $\mathscr{A}x$ is invariant under $\mathscr{A}$, we have $\mathscr{A}x = X$ and $x$ is strictly cyclic. (ii) Assume that $x \in N$ and $A \in \mathscr{A}$. Then $\mathscr{A}Ax \subset \mathscr{A}x$ and consequently $\mathscr{A}Ax \neq X$. That is, $Ax \in N$ for each $A$ in $\mathscr{A}$ which proves (ii). (iii) By (ii) $\mathscr{A}N \subset N$. Since $\mathscr{A}$ has a strictly cyclic vector, we know by Lemma 1 that $\bar{N}$ contains no strictly cyclic vector for $\mathscr{A}$. Thus by (i) $\bar{N}$ contains only noncyclic vectors for $\mathscr{A}$, which says that $N$ is closed. (iv) By (ii) $N$ is invariant under $\mathscr{A}$. By hypothesis $\mathscr{A}$ has a strictly cyclic vector so that $N \neq X$. These two observations essentially prove (iv) since an element $x$ of a proper invariant subset of $\mathscr{A}$ is necessarily an element of $N$. (v) If $N$ is nonlinear, then since $N$ is homogeneous, we know that $N \neq N + N$. Therefore, since $N + N$ is invariant under $\mathscr{A}$(by (ii) we know that $N + N = X$ by (iv)).

To see that there exist strictly cyclic operator algebras for which $N$ is linear and those for which $N$ is nonlinear let us reconsider Example 1 of [2].

EXAMPLE. Let $X$ be a Banach space, $\dim X \geqq 2$ and let $x_0 \in X$, $x_0 \neq 0$. Let each of $x^*$ and $y^*$ be a continuous linear functional on $X$ such that $x^*(x_0) = y^*(x_0) = 1$. For each $x$ in $X$ define $A_x$ by

$$A_x y = x^*(x)[y - y^*(y)x_0] + y^*(y)x$$

and let $\mathscr{A} = \{A_x : x \in X\}$.

It was observed in [2] that $\mathscr{A}$ is a strictly cyclic operator algebra with strictly cyclic, separating vector $x_0$.

A simple argument shows that a vector $y_0$ of $X$ is cyclic (and hence by Theorem 5 strictly cyclic) if and only if $y^*(y_0) \neq 0$ and $x^*(y_0) \neq 0$. Thus the set $N$ of noncyclic vectors coincides with $\ker y^* \cup \ker x^*$. Consequently, $N$ is linear if $x^*$ and $y^*$ are dependent and nonlinear otherwise.

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Received July 14, 1972 and in revised form August 25, 1972.

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