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RADICAL PROPERTIES INVOLVING ONE-SIDED IDEALS

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A radical P is called strongly right hereditary (srh) if $P(I) = I \cap P(R)$ for every right ideal I of each (not necessarily associative) ring R in a suitable universal class W. This is a one-sided version of the concept of a strongly hereditary radical class investigated by W.G. Leavitt and R.L. Tangeman. A discussion parallel to theirs is obtained including a construction of the minimal srh radical class in W containing a given class. Srh radicals are related to a new radical construction obtained by modifying the lower radical construction of Tangeman and D. Kreiling.

1. Introduction. A class M of not necessarily associative rings is called right hereditary if every right ideal of each ring in M is also in M. Subring hereditary classes are defined in a corresponding way. A universal class is a homomorphically closed, subring hereditary class of rings. A radical P of some universal class W is strongly hereditary if for all $R \in W$ we have $P(I) = I \cap P(R)$ for all ideals I of R, and strongly right hereditary (srh) if we have the same property for all right ideals I of R. Strongly hereditary radicals have been studied by W. G. Leavitt [4] and R. L. Tangeman [6] using the following property (a) which may be satisfied by a class M of rings in a universal class W:

(a) If $J \in M$ is an ideal of an ideal I of some $R \in W$, then the ideal J' of R generated by J is also in M. In § 2, we obtain a parallel discussion of srh radicals using the following modification of (a):

(ρ) If $J \in M$ is an ideal of a right ideal I of $R \in W$, then the ideal J' of R generated by J is also in M.

In a universal class W, the lower radical determined by a class M will be denoted by LM. In § 3, we introduce a new radical construction obtained by altering the construction of LM given by Tangeman and Kreiling [3] at the limit ordinal step. A brief summary of their construction may be found in [5], whose notation we will continue to use. Our construction is related to property (ρ) by Theorem 3.2.

For a class $M \subseteqq W$, the minimal right hereditary subclass of W containing M will be denoted by GM. Write $G_1M = M$ and, for $n \geqq 2$, $G_nM = \{R \in W: R$ is a right ideal of some ring in $G_{n-1}M\}$. Then $GM = \bigcup G_nM$, as in [5]. If $M = \{R\}$ consists of a single ring, we will omit braces and write, for example, $G_nM = G_nR$.

2. Srh radicals. The results of [4] and [6] all have one-sided

versions. In particular, following [4, Theorem 1], we have.

THEOREM 2.1. *A right hereditary radical class $P \subseteqq W$ is srh if and only if it has property* (ρ).

Next we show that property (ρ) is inherited by the lower radical. Our proof is an adaptation of an unpublished proof by Tangeman of [6, Theorem 2.4].

THEOREM 2.2. *Suppose $M \subseteqq W$ is homomorphically closed and has property (ρ). Then LM also satisfies (ρ).*

Proof. We will use the construction of LM due to Tangeman and Kreiling and the notation of [5]. By hypothesis $M_1 = M$ has property (ρ). Let $\beta > 1$ be an ordinal number and let J be an ideal of a right ideal I of a ring $R \in W$ such that $J \in M_\beta$. Let J' denote the ideal of R generated by J. Suppose the classes M_α satisfy (ρ) for all $\alpha < \beta$.

First suppose β is a limit ordinal. Then $J = \bigcup J_\gamma$, where $\{J_\gamma\}$ is a chain of ideals of J contained in $\bigcup_{\alpha<\beta} M_\alpha$. For each index γ, let K_γ be the ideal of I generated by J_γ. Then $J = \bigcup K_\gamma$. By property (ρ), each $K_\gamma \in \bigcup_{\alpha<\beta} M_\alpha$. Now let K'_γ be the ideal of R generated by K_γ. By (ρ) again, each $K'_\gamma \in \bigcup_{\alpha<\beta} M_\alpha$. Since $J' \supseteqq K_\gamma$ for each γ and J' is an ideal of R, $J' \supseteqq \bigcup K'_\gamma$. On the other hand, since $\bigcup K'_\gamma$ is an ideal of R containing $\bigcup K_\gamma = J$, we have $\bigcup K'_\gamma \supseteqq J'$. Hence $J' = \bigcup K'_\gamma \in M_\beta$.

If β is not a limit ordinal, then J has an ideal K with K, $J/K \in M_{\beta-1}$. Now if $P \subseteqq J$ is the ideal of I generated by K, then $P \in M_{\beta-1}$ by property (ρ). Moreover, $J/P \in M_{\beta-1}$ because J/P is a homomorphic image of J/K and $M_{\beta-1}$ is homomorphically closed [3, Lemma 2]. Now P generates an ideal Q of R with $Q \in M_{\beta-1}$ by the inductive hypothesis. The ideal of R/Q generated by $J + Q/Q$ is J'/Q. Since $P \subseteqq J \cap Q$, $J + Q/Q \simeq J/J \cap Q$ is a homomorphic image of J/P. Hence $J + Q/Q \in M_\beta$ and so, using (ρ) again, $J'/Q \in M_{\beta-1}$. Since Q, $J'/Q \in M_{\beta-1}$, we have $J' \in M_\beta$. The theorem follows by transfinite induction.

Let $EM = \{J' : J$ is an ideal of a right ideal of a ring $R \in W$, $J \in M$, and J' is the ideal of R generated by $J\}$. The homomorphic closure of M will be denoted by HM. We have the following one-sided version of [6, Corollary to Theorem 2.5].

THEOREM 2.3. *If W is a universal class and $M \subseteqq W$, then there exists a unique minimal radical class in W containing M and satisfying property* (ρ).

Proof. Let $M_1^* = EHM$ and, inductively, $M_n^* = EHM_{n-1}^*$ for all integers $n > 1$. Then $M^* = UM_n^*$ is a homomorphically closed class satisfying (ρ), so that LM^* satisfies (ρ) by Theorem 2.2. On the other hand, any radical class which satisfies property (ρ) and contains M may be seen by induction to contain M^* and hence LM^*.

EXAMPLE 2.1. The class LM^* need not be hereditary when M is hereditary. For let $K = GF(2)$ and let $R = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}: a, b \in K\right\}$. We identify isomorphic rings; thus $K \simeq \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}: a \in K\right\}$ is a right ideal of R with $K' = R$. R has the ideal $I = \left\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}: b \in K\right\}$. Let $W = \{R, K, I, 0\}$, $M = \{K, 0\}$. Then M is hereditary, while $LM^* = M^* = \{R, K, 0\}$ is not.

As in [6, Corollary (c)] and [8, Corollary 2.7], we also have

THEOREM 2.4. *If W is a universal class and $M \subseteq W$, then there is a unique minimal srh radical class in W containing M.*

Proof. Define $\bar{M}_1 = EGHM$ and, for all $n > 1$, $\bar{M}_n = EGH\bar{M}_{n-1}$. Then $\bar{M} = \bigcup \bar{M}_n$ is the minimal subclass of W which contains M, satisfies property (ρ) and is right hereditary and homomorphically closed. Then $L\bar{M}$ is the desired srh radical class, for by Theorem 2.2 $L\bar{M}$ has property (ρ) and by [5, Theorem 2], $L\bar{M}$ is right hereditary. Thus by Theorem 2.1, $L\bar{M}$ is srh; it is again easy to see that $L\bar{M}$ is minimal.

We turn to a consideration of two properties similar to property(ρ).

THEOREM 2.5. *Let M be a class of rings satisfying property (ρ). For all $R \in W$, if $I \in M$ is in GR, then the ideal I' of R generated by I is also in M.*

Proof. The theorem is trivially satisfied when $I \in M \cap G_1R$. Thus for induction assume for all $R \in W$ and all $I \in M \cap G_nR$ that $I' \in M$, where I' is the ideal of R generated by I. Let $K \in M \cap G_{n+1} R$ so that $K \in M \cap G_nJ$ for some right ideal J of R. By induction $K^* \in M$ where K^* is the ideal of J generated by K. But then (ρ) implies $K' = K^{*\prime} \in M$.

COROLLARY. *Property (ρ) is equivalent to the following property (ρ'):*

(ρ') *If $J \in M$ is a right ideal of a right ideal I of $R \in W$, then the ideal J' of R generated by J is also in M.*

Consider the following property (σ): If $J \in M$ is a right ideal of R, then the ideal of J' of R generated by J is also in M. In general this property is not inherited by LM as may be seen from the following example (for which we thank the referee).

EXAMPLE 2.2. Let K be generated over $GF(2)$ by x, y, z where $x^2 = y^2 = 0$, $xy = yx = x$, $yz = xz = zy = y$, and $zx = z^2 = z$. Then $I = \{0, x\}$ is an ideal of $R = \{0, x, y, x + y\}$ and R is the only proper right ideal of K. Also K is simple so that $R' = K$. For the universal class W consisting of K and all its subrings, the class $M = \{0, I\}$ has property (σ). However, LM does not have the property since $R \in LM$ whereas $R' = K \notin LM$.

For semisimple classes, we have the following one-sided version of [1, Theorem 4.1] and [6, Theorem 3.1], which we state without proof.

THEOREM 2.6. *Q is a semisimple class for a radical class P with property (ρ) if and only if Q has properties* (b), (c), *and* (d) *of* [1, *Theorem* 4.1] *and is right hereditary.*

In general it cannot be expected that semisimple subideals will generate semisimple ideals, as in property (a). Indeed, if the radical class is not hereditary, a semisimple subideal may even generate a radical ideal. We give two examples using well-known radicals in the universal class of associative rings.

EXAMPLE 2.3. Let A be a ring isomorphic to the ring of even integers with generator a. Let $B = \{0, x\}$, $C = \{0, y\}$ be zero rings of order two. Let $I = A \oplus B$, and form R by adjoining C to I in such a way that the additive group of R is $I + C$ (direct sum), $(na)y = y(na) = nx$ for all integers n, and $xy = yx = 0$. Then I is an ideal of R and A is a nil-semisimple ideal of I, but $A' = I$ has the nil ideal B.

EXAMPLE 2.4. Let A be the zero ring whose additive group is $Z_p(\infty)$ and let B be the ring of polynomials of degree $\geqq 1$ over $GF(2)$. Define the commutative ring R as follows. The additive group of R is the direct sum $A + B$; the multiplication within A and B is as usual, and we define $(a/p^n)x^i = a/p^{n+i}$, extending this multiplication to R in the natural way.

Let I be the subring of A of order p. Thus I is an ideal of A, and the ideal I generates in R is A. In the upper radical of the class of all simple rings (see [2, page 14]), I is semisimple and A is radical.

3. Radical constructions involving one-sided ideals. Let M be any class contained in a universal class W. We will construct a class ΔM (depending of course on the universal class W) by modifying the radical construction of [3]. Briefly, let $\Delta_1 M$ be the homomorphic closure of M. We proceed inductively to define a class $\Delta_\beta M$ for each ordinal number β. If $\beta - 1$ exists, let $\Delta_\beta M = \{R \in W$: R has an ideal J such that $J, R/J \in \Delta_{\beta-1} M\}$. If β is a limit ordinal, define $R \in$

$\Delta_\beta M$ if and only if R is the union of a chain $\{I_\gamma\}$ of *right* ideals of R such that each $I_\gamma \in \bigcup_{\alpha<\beta} \Delta_\alpha M$. Finally, let $\Delta M = \bigcup_\beta \Delta_\beta M$.

By modifying suitably the proof of [3, Theorem 2] we have

THEOREM 3.1. *ΔM is a radical class.*

The corresponding construction using *left* ideals yields a radical class we will call ΛM.

THEOREM 3.2. *If M is homomorphically closed and has property (ρ), then $LM = \Delta M$.*

Proof. Since $M \subseteqq \Delta M$ and ΔM is radical, $LM \subseteqq \Delta M$. Thus assume for induction that, for β a given ordinal, $\Delta_\propto M \subseteqq LM$ for all $\propto < \beta$. If $R \in M_\beta$ is a nonlimit ordinal then $I, R/I \in \Delta_{\beta-1} M \subseteqq LM$, so that $R \in LM$. If β is a limit ordinal then $R = I_\gamma$ for some chain $\{I_\gamma\}$ of right ideals contained in $\bigcup_{\alpha<\beta} \Delta_\propto M \subseteqq LM$. But by Theorem 2.2, LM has property (ρ). Thus if I'_γ is the ideal of R generated by I_γ, then $I'_\gamma \in LM$ and so $R = \bigcup I'_\gamma \in LM$. Thus $\Delta_\beta M \subseteqq LM$ and so $\Delta M \subseteqq LM$.

This is not a necessary condition, for let M be the nil radical class in the universal class of associative rings. Then $M = LM = \Delta M$, but M does not have property (ρ) by Theorem 2.6 because the nil-semisimple rings do not form a right hereditary class.

Even in the associative case, ΔM and ΛM may be unequal.

EXAMPLE 3.1. Let R be the associative algebra over the field $GF(2)$ generated by a countable number of symbols $\{x_i: i = 1, 2, \cdots\}$ subject to the relations $x_i x_j = x_j$ for all i, j. For each n, let I_n be the left ideal generated by $\{x_1, \cdots, x_n\}$. Then $M = \{I_n: n = 1, 2, \cdots\}$ is a chain of left ideals of R and $R = I_n$, so that $R \in \Lambda M$. Since R has no proper right ideals and $R \notin M_1 = \Delta_1 M$, R cannot be in ΔM.

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