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BOUNDS FOR PRODUCTS OF INTERVAL FUNCTIONS

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Since it is possible for ${}_a\Pi^b(1 + G)$ to exist and not be zero when G is unbounded and $1 + G$ is not bounded away from zero, the conditions under which products of the form $|\Pi_1^n[1 + G(x_{q-1}, x_q)]|$ are bounded or bounded away from zero for suitable subdivisions $\{x_q\}_0^n$ of $[a, b]$ are important in many theorems concerning product integrals. Conditions are obtained for such bounds to exist for products of the form $\Pi(1 + FG)$ and $\Pi(1 + F + G)$, where F and G are functions from $R \times R$ to R. Further, these results are used to obtain an existence theorem for product integrals.

All integrals and definitions are of the subdivision-refinement type, and functions are from the subset $\{(x, y): x < y\}$ of $R \times R$ to R, where R represents the set of real numbers. If $D = \{x_q\}_0^n$ is a subdivision of $[a, b]$ and G is a function, then $D(I) = \{[x_{q-1}, x_q]\}_1^n$ and $G_q = G(x_{q-1}, x_q)$. The statements that G is bounded, $G \in OP^\circ$, $G \in OQ^\circ$ and $G \in OB^\circ$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and a positive number B such that if $J = \{x_q\}_0^n$ is a refinement of D, then

(1) $|G(u)| < B$ for $u \in J(I)$,

(2) $|\Pi_r^s(1 + G_q)| < B$ for $1 \leqq r \leqq s \leqq n$,

(3) $|\Pi_r^s(1 + G_q)| > B$ for $1 \leqq r \leqq s \leqq n$, and

(4) $\Sigma_{J(I)}|G| < B$,

respectively. The notation $\{x_{qr}\}_0^{n(q)}$ represents a subdivision of an interval $[x_{q-1}, x_q]$ defined by a subdivision $\{x_q\}_0^n$. If G is a function, then $G \in S_1$ on $[a, b]$ only if $\lim_{x,y\to p} + G(x, y)$ and $\lim_{x,y\to p} - G(x, y)$ exist and are zero for $p \in [a, b]$, and $G \in S_2$ on $[a, b]$ only if $\lim_{x\to p} + G(p, x)$ and $\lim_{x\to p} - G(x, p)$ exist for $p \in [a, b]$. Further, $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b \left|G - \int G\right| = 0$, and $G \in OM^\circ$ on $[a, b]$ only if ${}_x\Pi^y(1 + G)$ exists for $a \leqq x < y \leqq b$ and $\int_a^b |1 + G - \Pi(1 + G)| = 0$. Also, $G \in OQ^1$ and $G \in OB^*$ on $[a, b]$ if there exists a subdivision $D = \{x_q\}_0^n$ of $[a, b]$ such that

(1) if $1 \leqq q \leqq n$ and $x_{q-1} < x < y < x_q$, then $G \in OQ^\circ$ on $[x, y]$, and

(2) if $1 \leqq q \leqq n$, then either $G \in OB^\circ$ on $[x_{q-1}, x_q]$ or $G - 1 \in OB^\circ$ on $[x_{q-1}, x_q]$,

respectively. The statement that G is almost bounded above by β (or, almost bounded below by β) on $[a, b]$ means there exists a positive integer N such that if D is a subdivision of $[a, b]$ and $u \in H$ only if $u \in D(I)$ and $G(u) > \beta$ (or, $G(u) < \beta$) then H has less than N elements. Consult B. W. Helton [2] and J. S. MacNerney [4] for additional details.

THEOREM 1. *If G is a function, then the following are equivalent:*
(1) $G \in OB^\circ$ *on* $[a, b]$, *and*
(2) *if* $F \in OP^\circ$ *on* $[a, b]$, *then* $F + G \in OP^\circ$ *on* $[a, b]$.

Proof $(2 \rightarrow 1)$. Let F be the function such that $F(x, y) = 0$ if $G(x, y) \geqq 0$ and $F(x, y) = -2$ if $G(x, y) < 0$. Hence, if J is a subdivision of $[a, b]$, then

$$|\Pi_{J(I)}(1 + F + G)| = \Pi_{J(I)}(1 + |G|) ,$$

which can be bounded only if $G \in OB^\circ$.

Proof $(1 \rightarrow 2)$. Suppose $F \in OP^\circ$. There exist positive numbers B and C with $B > 1$, a positive integer i and a subdivision D of $[a, b]$ such that if $J = \{x_q\}_0^w$ is a refinement of D, then
(1) $|\Pi_r^s(1 + F_q)| < B$ for $1 \leqq r \leqq s \leqq w$,
(2) $\exp[4B\,\Sigma_{J(I)}|G|] < C$,
(3) if T is a collection of nonintersecting subsets of $J(I)$, then the number of $t \in T$ such that $\exp[4B\Sigma_t|G|] > 2$ is less than i, and
(4) the number of $u \in J(I)$ such that $|G(u)| > 1/4B$ is less than i.
Let $J = \{x_q\}_0^w$ be a refinement of D and suppose $1 \leqq r \leqq s \leqq w$. Let $L = \{[x_{q-1}, x_q]\}_r^s$, and let H be the subset of L such that $u \in H$ only if $|1 + F(u)| \leqq 1/4B$. Further, let K be the collection of subsets of L such that $k \in K$ only if there exist $u, v \in H$ such that u precedes v on $[a, b]$ and either
(1) $k = \{t \mid t$ precedes v and follows $u\}$ and $k \cap H = \varnothing$,
(2) u is the first element in H and $k = \{t \mid t$ precedes $u\}$, or
(3) v is the last element in H and $k = \{t \mid t$ follows $v\}$.
Let $u \in M$ only if $u \in H$ and $|G(u)| > 1/4B$, and let $k \in N$ only if $k \in K$ and $\exp[4B\,\Sigma_k|G|] > 2$. Hence, M and N each has less than i elements. Also, K has at most one more element than H. Hence, $K - N$ can have at most i more elements than $H - M$. Let j, m and n denote the number of elements in M, $H - M$ and $K - N$, respectively, and suppose $U = \bigcup_{k \in K} k$. Hence,

$$\begin{aligned}
&|\Pi_L(1 + F + G)| \\
&\quad \leqq \{\Pi_H[|1 + F| + |G|]\} \cdot \{|\Pi_U(1 + F + G)|\} \\
&\quad \leqq \{\Pi_M[1/4B + |G|]\} \cdot \{\Pi_{H-M}[1/4B + |G|]\} \cdot \{|\Pi_U(1 + F + G)|\} \\
&\quad \leqq \{(1/4B)^j C\} \cdot \{1/4B + 1/4B\}^m \cdot \{|\Pi_U(1 + F + G)|\} \\
&\quad \leqq C\{1/2B\}^m \cdot \{\Pi_{k \in K}|\Pi_k[1 + F][1 + (1 + F)^{-1}G]|\} \\
&\quad \leqq C\{1/2B\}^m \cdot \{\Pi_{k \in K}[|\Pi_k(1 + F)|][\Pi_k(1 + 4B|G|)]\} \\
&\quad = C\{1/2B\}^m \cdot \{\Pi_{k \in N}[|\Pi_k(1 + F)|][\Pi_k(1 + 4B|G|)]\} .
\end{aligned}$$

$$\{\Pi_{k\in K-N}[|\Pi_k(1+F)|][\Pi_k(1+4B|G|)]\}$$
$$\leqq C\{1/2B\}^m \cdot \{BC\}^i \cdot \{2B\}^n$$
$$= B^i C^{i+1}(2B)^{n-m} \leqq B^i C^{i+1}(2B)^i \, .$$

LEMMA 1.1. *If* $\int_a^b F$ *exists, then* $F \in OA^\circ$ *on* $[a, b]$.

This result is due to A. Kolmogoroff [3, p. 669]. Further, related results have also been obtained by W. D. L. Appling [1, Th. 2, p. 155] and B. W. Helton [2, Th. 4.1, p. 304].

COROLLARY 1.1. *If* $\int_a^b F$ *exists, then the following are equivalent:* (1) $F \in OP^\circ$ *on* $[a, b]$, *and* (2) $\int F \in OP^\circ$ *on* $[a, b]$.

Indication of proof. Since $\int_a^b F$ exists, $F \in OA^\circ$ [Lemma 1.1]. The result now follows by using Theorem 1.

COROLLARY 1.2. *If* $F \in OP^\circ$ *on* $[a, b]$, ${}_a\Pi^b(1 + F)$ *exists and* $\int_a^b |G| = 0$, *then* ${}_a\Pi^b(1 + F + G)$ *exists and is* ${}_a\Pi^b(1 + F)$.

Indication of proof. A related result is proved by B. W. Helton [2, Th. 5.6, p. 315]. This result follows by an argument similar to the one used in that theorem since Theorem 1 implies that $F + G \in OP^\circ$.

COROLLARY 1.3. *If* G *is a function, then the following are equivalent:*

(1) $G \in OP^\circ$ *on* $[a, b]$, *and*

(2) *if* $F \in OB^\circ$ *on* $[a, b]$, *then* $F + G \in OP^\circ$ *on* $[a, b]$.

Proof. Theorem 1 establishes that (1) implies (2). Further, (2) implies (1) since $F \equiv 0$ belongs to OB°.

B. W. Helton has shown if G is a function from $S \times S$ to N such that $G \in OA^\circ$ and $G \in OB^\circ$, then $G \in OM^\circ$, where S represents a linearly ordered set and N represents a ring which has a multiplicative identity element denoted by 1 and has a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$ [2, Th. 3.4 ($1 \rightarrow 2$), p. 301]. We now use Theorem 1 to establish a related result. In particular, we show that if F and G are functions from $R \times R$ to R such that $F \in OM^\circ$, $F \in OP^\circ$, $F \in S_1 \cap S_2$ and $G \in OB^\circ$ on $[a, b]$ and $\int_a^b G$ exists, then $F + G \in OM^\circ$ on $[a, b]$.

LEMMA 2.1. *If* F *and* G *are functions such that* $F \in OM^\circ, F \in$

OP°, $F \in S_1$ and $G \in OB^\circ$ on $[a, b]$ and $\varepsilon > 0$, then there exists a subdivision $\{y_q\}_0^u$ of $[a, b]$ such that if $y_{q-1} < x < y < y_q$ and H is a subdivision of $[x, y]$, then

$$|1 - \Pi_{H(I)}(1 + F + G)| < \varepsilon .$$

Further, if $F \in S_2$ and $G \in S_2$ on $[a, b]$, then there exists a subdivision $\{z_q\}_0^v$ of $[a, b]$ such that if $z_{q-1} \leqq x < y \leqq z_q$ and H is a subdivision of $[x, y]$, then

$$|1 + F(x, y) + G(x, y) - \Pi_{H(I)}(1 + F + G)| < \varepsilon .$$

Proof. Suppose F and G are functions such that $F \in OM^\circ$, $F \in OP^\circ$, $F \in S_1$ and $G \in OB^\circ$ on $[a, b]$ and $\varepsilon > 0$. It follows from Theorem 1 that $F + G \in OP^\circ$. There exist a subdivision $D_1 = \{y_q\}_0^u$ of $[a, b]$ and a number $B > 1$ such that if $J = \{x_q\}_0^n$ is a refinement of D_1, then

(1) $|\Pi_i^j(1 + F_q)| < B$ and $|\Pi_i^j(1 + F_q + G_q)| < B$ for $1 \leqq i \leqq j \leqq n$,

(2) $|F(x, y)| < \varepsilon/9B$ and $\Sigma_{H(I)}|G| < \varepsilon/9B^3$ if $1 \leqq q \leqq n$, $x_{q-1} < x < y < x_q$ and H is a subdivision of $[x, y]$, and

(3) $\Sigma_q|(1 + F_q) - \Pi_{H_q(I)}(1 + F)| < \varepsilon/9B$, where H_q is a subdivision of $[x_{q-1}, x_q]$ for $q = 1, 2, \cdots, n$.

Suppose $1 \leqq q \leqq u$ and $y_{q-1} < x < y < y_q$. If $H = \{h_q\}_0^r$ is a subdivision of $[x, y]$, then

$$\begin{aligned}
&|1 - \Pi_{H(I)}(1 + F + G)| \\
&\quad = |1 + F(x, y) - F(x, y) - \{\Pi_{q=1}^r(1 + F_q) \\
&\qquad + \Sigma_{q=1}^r[\Pi_{j=1}^{q-1}(1 + F_j)][G_q][\Pi_{k=q+1}^r(1 + F_k + G_k)]\}| \\
&\quad \leqq |1 + F(x, y) - \Pi_{q=1}^r(1 + F_q)| + |F(x, y)| \\
&\qquad + \Sigma_{q=1}^r|\Pi_{j=1}^{q-1}(1 + F_j)||G_q||\Pi_{k=q+1}^r(1 + F_k + G_k)| \\
&\quad < \varepsilon/9B + \varepsilon/9B + B^2\varepsilon/9B^3 = \varepsilon/3B < \varepsilon .
\end{aligned}$$

We now make the additional suppositions that $F \in S_2$ and $G \in S_2$ on $[a, b]$. There exists a subdivision $E = \{w_q\}_0^{2u+1}$ of $[a, b]$ such that

(1) $y_q \in (w_{2q}, w_{2q+1})$ for $1 \leqq q < u$,

(2) $|F(y_q, w_{2q+1}) + G(y_q, w_{2q+1}) - F(y_q, x) - G(y_q, x)| < \varepsilon/2$ for $0 \leqq q < u$ and $x \in (y_q, w_{2q+1}]$, and

(3) $|F(w_{2q}, y_q) + G(w_{2q}, y_q) - F(x, y_q) - G(x, y_q)| < \varepsilon/2$ for $0 < q \leqq u$ and $x \in [w_{2q}, y_q)$.

Let $D_2 = \{z_q\}_0^{3u}$ be the subdivision $D_1 \cup E$ of $[a, b]$. Suppose $1 \leqq q \leqq 3u$, $z_{q-1} \leqq x < y \leqq z_q$ and H is a subdivision of $[x, y]$. If either $z_{q-1} < x < y < z_q$ or neither z_{q-1} nor z_q is in D_1, then

$$\begin{aligned}
&|1 + F(x, y) + G(x, y) - \Pi_{H(I)}(1 + F + G)| \\
&\quad \leqq |F(x, y)| + |G(x, y)| + |1 - \Pi_{H(I)}(1 + F + G)| \\
&\quad < \varepsilon/9B + \varepsilon/9B^3 + \varepsilon/3B < \varepsilon .
\end{aligned}$$

If $z_{q-1} \in D_1$, $x = z_{q-1}$ and $H = \{h_q\}_0^r$, then

$$\begin{aligned}
&|1 + F(x, y) + G(x, y) - \Pi_{H(I)}(1 + F + G)| \\
&\quad \leqq |F(x, y) + G(x, y) - F(x, h_1) - G(x, h_1)| \\
&\qquad + |1 + F(x, h_1) + G(x, h_1)|\,|1 - \Pi_2^r[1 + F(h_{q-1}, h_q) \\
&\qquad + G(h_{q-1}, h_q)]| \\
&\quad < \varepsilon/2 + B\varepsilon/3B < \varepsilon\,.
\end{aligned}$$

If $z_q \in D_1$ and $y = z_q$, the necessary inequality follows in a similar manner. Therefore, D_2 is the desired subdivision.

THEOREM 2. *If F and G are functions such that $F \in OM^\circ$, $F \in OP^\circ$, $F \in S_1 \cap S_2$ and $G \in OB^\circ$ on $[a, b]$ and $\int_a^b G$ exists, then $F + G \in OM^\circ$ on $[a, b]$.*

Proof. We initially show that if $\varepsilon > 0$ then there exists a subdivision D of $[a, b]$ such that if $H = \{x_q\}_0^n$ is a refinement of D and H_q is a subdivision of $[x_{q-1}, x_q]$ for $q = 1, 2, \cdots, n$, then

$$\Sigma_1^n |1 + F_q + G_q - \Pi_{H_q(I)}(1 + F + G)| < \varepsilon\,.$$

Let $\varepsilon > 0$. It follows from Lemma 1.1 that $G \in OA^\circ$ and from Theorem 1 that $F + G \in OP^\circ$. Thus, by employing the hypothesis and Lemma 2.1, there exist a subdivision $D_1 = \{y_q\}_0^u$ of $[a, b]$ and a number $B > 1$ such that if $J = \{x_q\}_0^n$ is a refinement of D_1, then

(1) $\Sigma_{J(I)}|G| < B$,

(2) $|\Pi_i^j(1 + F_q)| < B$ for $1 \leqq i \leqq j \leqq n$,

(3) $\Sigma_1^n |G_q - \Sigma_{L_q(I)} G| < \varepsilon/5$ and $\Sigma_1^n |(1 + F_q) - \Pi_{L_q(I)}(1 + F)| < \varepsilon/5$,

where L_q is a subdivision of $[x_{q-1}, x_q]$ for $1 \leqq q \leqq n$, and

(4) $|1 - \Pi_{H(I)}(1 + F)| < \varepsilon/5B$ and $|1 - \Pi_{H(I)}(1 + F + G)| < \varepsilon/5B^2$ for $1 \leqq q \leqq n$, $x_{q-1} < x < y < x_q$ and H a subdivision of $[x, y]$.

Further, it also follows from Lemma 2.1 that there exists a subdivision $D_2 = \{z_q\}_0^v$ of $[a, b]$ such that if $1 \leqq q \leqq v$, $z_{q-1} \leqq x < y \leqq z_q$ and H is a subdivision of $[x, y]$, then

$$|1 + F(x, y) + G(x, y) - \Pi_{H(I)}(1 + F + G)| < \varepsilon/10u\,.$$

Let $D = D_1 \cup D_2$, and suppose $H = \{x_q\}_0^n$ is a refinement of D and $H_q = \{x_{qr}\}_0^{n(q)}$ is a subdivision of $[x_{q-1}, x_q]$ for $1 \leqq q \leqq n$. Let P be the set such that $q \in P$ only if $[x_{q-1}, x_q]$ has an end point in D_1, and let $Q = \{i\}_1^n - P$. Further, to simplify notation, let $F_{qr} = F(x_{q,r-1}, x_{qr})$, $G_{qr} = G(x_{q,r-1}, x_{qr})$, $A_{qr} = \Pi_{j=1}^{r-1}(1 + F_{qj})$ and $B_{qr} = \Pi_{k=r+1}^{n(q)}(1 + F_{qk} + G_{qk})$. Thus,

$$\begin{aligned}
&\Sigma_{q=1}^n |1 + F_q + G_q - \Pi_{H_q(I)}(1 + F + G)| \\
&\quad \leqq \Sigma_{q \in P}|1 + F_q + G_q - B_{q0}|
\end{aligned}$$

$$\begin{aligned}
&+ \Sigma_{q\in Q}|1 + F_q + G_q - B_{q0}| \\
&< 2u\varepsilon/10u + \Sigma_{q\in Q}|1 + F_q + G_q - [A_{q,n(q)+1} \\
&\quad + \Sigma_{r=1}^{n(q)} A_{qr}G_{qr}B_{qr}]| \\
&\leqq \varepsilon/5 + \Sigma_{q\in Q}|1 + F_q - A_{q,n(q)+1}| \\
&\quad + \Sigma_{q\in Q}|G_q - \Sigma_{r=1}^{n(q)} A_{qr}G_{qr}B_{qr}| \\
&< 2\varepsilon/5 + \Sigma_{q\in Q}|G_q - \Sigma_{r=1}^{n(q)} G_{qr}| \\
&\quad + \Sigma_{q\in Q}|\Sigma_{r=1}^{n(q)} G_{qr} - \Sigma_{r=1}^{n(q)} A_{qr}G_{qr}B_{qr}| \\
&< 3\varepsilon/5 + \Sigma_{q\in Q}\Sigma_{r=1}^{n(q)}|1 - A_{qr}||G_{qr}| \\
&\quad + \Sigma_{q\in Q}\Sigma_{r=1}^{n(q)}|A_{qr}||G_{qr}||1 - B_{qr}| \\
&< 3\varepsilon/5 + (\varepsilon/5B)B + (\varepsilon/5B^2)B^2 = \varepsilon .
\end{aligned}$$

Hence, if $a \leqq x < y \leqq b$ and $\varepsilon > 0$, then there exist a subdivision D of $[a, b]$ and a number B such that if $H = \{x_q\}_0^n$ is a refinement of D and H_q is a subdivision of $[x_{q-1}, x_q]$, then

(1) $|\Pi_i^j(1 + F_q + G_q)| < B$ for $1 \leqq i \leqq j \leqq n$, and

(2) $\Sigma_1^n|1 + F_q + G_q - \Pi_{H_q(I)}(1 + F + G)| < \varepsilon/B^2$.

Thus, if H and H_q are defined as above, then

$$\begin{aligned}
&|\Pi_1^n(1 + F_q + G_q) - \Pi_1^n\Pi_{H_q(I)}(1 + F + G)| \\
&\quad \leqq B^2\Sigma_1^n|1 + F_q + G_q - \Pi_{H_q(I)}(1 + F + G)| \\
&\quad < B^2(\varepsilon/B^2) = \varepsilon .
\end{aligned}$$

Therefore, ${}_x\Pi^y(1 + F + G)$ exists.

It now follows that $\int_a^b|1 + F + G - \Pi(1 + F + G)| = 0$. Hence, $F + G \in OM^\circ$ on $[a, b]$.

THEOREM 3. *If $F \in OQ^\circ$, $G \in OB^\circ$ and $1 + F + G$ is bounded away from zero on $[a, b]$, then $F + G \in OQ^\circ$ on $[a, b]$.*

Proof. There exist a subdivision D of $[a, b]$, a positive number $c < 1$ and a positive integer m such that if $J = \{x_q\}_0^n$ is a refinement of D, then

(1) $|1 + F_q + G_q| > c$ for $1 \leqq q \leqq n$,

(2) $|\Pi_i^j(1 + F_q)| > c$ for $1 \leqq i \leqq j \leqq n$, and

(3) if K is any collection of nonintersecting subsets of $J(I)$, then the number of $k \in K$ such that $\Sigma_k|G|/c > 1/2$ is less than m.

Suppose $J = \{x_q\}_0^n$ is a refinement of D and $1 \leqq r \leqq s \leqq n$. Let $K = \{k_j\}$ be the collection of nonintersecting subsets of $\{[x_{q-1}, x_q]\}_r^s$ such that

(1) $k_1 = \{[x_{q-1}, x_q]\}_{m(1)}^{n(1)}$, where $m(1)$ is the first integer such that $m(1) \geqq r$ and $|G_{m(1)}|/c \leqq 1/2$ and $n(1)$ is the largest integer such that $n(1) < s$, $\Sigma_{m(1)}^{n(1)}|G_q|/c \leqq 1/2$ and $\Sigma_{m(1)}^{n(1)+1}|G_q|/c > 1/2$ if such an integer

exists and s otherwise, and

(2) $k_j = \{[x_{q-1}, x_q]\}_{m(j)}^{n(j)}$, where $m(j)$ is the first integer such that $m(j) > n(j-1)$ and $|G_{m(j)}|/c < 1/2$ and $n(j)$ is the largest integer such that $n(j) \leqq s$, $\Sigma_{m(j)}^{n(j)} |G_q|/c \leqq 1/2$ and $\Sigma_{m(j)}^{n(j)+1} |G_q|/c > 1/2$ if such an integer exists and s otherwise.

Let $U = \bigcup_{k \in K} k$ and $V = \{[x_{q-1}, x_q]\}_r^s - U$. Note that K and V each has a maximum of m elements. Thus,

$$\begin{aligned}
&|\Pi_r^s(1 + F_q + G_q)| \\
&\quad = \{\Pi_V |1 + F + G|\}\{\Pi_U |1 + F + G|\} \\
&\quad \geqq c^m \Pi_U[|1 + F| - |G|] \\
&\quad = c^m \Pi_{k \in K}\{\Pi_k |1 + F|\}\{\Pi_k[1 - |G|(|1 + F|)^{-1}]\} \\
&\quad \geqq c^{2m} \Pi_{k \in K}\{\Pi_k(1 - |G|/c)\} \\
&\quad \geqq c^{2m} \Pi_{k \in K}[1 - \Sigma_k |G|/c] \geqq c^{2m}/2^m \, .
\end{aligned}$$

COROLLARY 3.1. *If* $\int_a^b F$ *exists, then the following are equivalent*: (1) $F \in OQ^\circ$ *on* $[a, b]$, *and* (2) $\int F \in OQ^\circ$ *on* $[a, b]$.

Indication of proof. Since $\int_a^b F$ exists, $F \in OA^\circ$ [Lemma 1.1]. The result now follows by using Theorem 3.

COROLLARY 3.2. *If* G *is a function, then the following are equivalent*: (1) $G \in OQ^1$ *on* $[a, b]$, *and* (2) *if* $F \in OB^\circ$ *on* $[a, b]$, *then* $F + G \in OQ^1$ *on* $[a, b]$.

Indication of proof. Since $F \equiv 0$ is in OB°, (2) implies (1). Further, it follows from Theorem 3 that (1) implies (2).

LEMMA 3.1. *If* $0 \leqq G \leqq 1$ *and* $G \notin OB^\circ$ *on* $[a, b]$, *then* $-G \notin OQ^\circ$ *on* $[a, b]$.

Indication of proof. If H is a subdivision of $[a, b]$, then

$$\begin{aligned}
\Pi_{H(I)}(1 - G) &= \exp[\Sigma_{H(I)} \ln(1 - G)] \\
&= \exp[-\Sigma_{H(I)}\Sigma_1^\infty G^i/i] \, .
\end{aligned}$$

Thus, $\Pi_{H(I)}(1 - G) \to 0$ as $\Sigma_{H(I)} G \to \infty$.

COROLLARY 3.3. *If* G *is a function, then the following are equivalent*: (1) $G \in OB^\circ$ *on* $[a, b]$, *and* (2) *if* $F \in OQ^1$ *on* $[a, b]$, *then* $F + G \in OQ^1$ *on* $[a, b]$.

Proof. Since it follows from Theorem 3 that (1) implies (2), we need only show that (2) implies (1). The function $|G|$ is almost bounded above on $[a, b]$ by 1/2. If this is not so, then a contradiction follows by considering the function F such that

(1) $F(x, y) = 0$ if $-1/2 \leqq G(x, y) \leqq 0$,
(2) $F(x, y) = -G(x, y) - 1/2$ if $G(x, y) < -1/2$,
(3) $F(x, y) = -2$ if $0 < G(x, y) \leqq 1/2$, and
(4) $F(x, y) = -G(x, y) - 3/2$ if $G(x, y) > 1/2$.

Thus, although $F \in OQ^1$, $F + G \notin OQ^1$ since $|1 + F + G| \leqq 1$ and the number of intervals for which $|1 + F + G| = 1/2$ is unbounded. Now, if $G \notin OB^\circ$, a contradiction follows from Lemma 3.1 by using the function F such that

(1) $F(x, y) = -2$ if $G(x, y) \geqq 0$, and
(2) $F(x, y) = 0$ if $G(x, y) < 0$.

THEOREM 4. *If G is a function, then the following are equivalent:*

(1) *if* $\int_a^b |F| = 0$, *then* $FG \in OB^\circ$,

(2) *if* $\int_a^b |F| = 0$, *then* $FG \in OP^\circ$,

(3) *if* $\int_a^b |F| = 0$, *then* $FG \in OQ^\circ$, *and*

(4) *G is bounded on* $[a, b]$.

Proof. It follows readily that (4) implies (1). Further, it follows that (4) implies (2) and (3) by using Theorems 1 and 3, respectively. If $G(x, y)$ as $x, y \to p^-$, $G(x, y)$ as $x, y \to p^+$, $G(x, p)$ as $x \to p^-$ and $G(p, x)$ as $x \to p^+$ are bounded for each $p \in [a, b]$, then it follows from the covering theorem that G is bounded on $[a, b]$. If one or more of these bounds fail to exist for some $p \in [a, b]$, then there exists a sequence $\{(y_q, z_q)\}_1^\infty$ of distinct subintervals of $[a, b]$ such that $|G(y_q, z_q)| > q^3$ for $q = 1, 2, \cdots$, and if $\{x_q\}_0^n$ is a subdivision of $[a, b]$ and r is a positive integer then there exist positive integers i and j such that $j > r$ and $x_{i-1} \leqq y_j < z_j \leqq x_i$. Contradictions to (1) and (2) now follow by considering the function F such that

$$F(x, y) = [G(x, y)]/[q^2 |G(x, y)|]$$

if there exists a positive integer q such that $x = y_q$ and $y = z_q$ and $F(x, y) = 0$ otherwise. Here $\int_a^b |F| = 0$, but FG is in neither OB° nor OP°. Further, a contradiction to (3) follows by considering the function F such that $F(x, y) = [-G(x, y)]^{-1}$ if there exists a positive integer q such that $x = y_q$ and $y = z_q$ and $F(x, y) = 0$ otherwise.

LEMMA 5.1. *If G is a function such that*

(1) *G is almost bounded above by 1/3 on $[a, b]$, and*

(2) *if $F \in OP^\circ$ on $[a, b]$, then $FG \in OP^\circ$ on $[a, b]$,*

then $G \in OB^\circ$ on $[a, b]$.

Proof. Suppose $G \notin OB^\circ$ on $[a, b]$. It follows from Theorem 4 that G is bounded on $[a, b]$. There exists a set $\{C(i)\}_1^\infty$ such that

(1) $C(i)$ is a finite set of nonoverlapping subintervals of $[a, b]$ which can be grouped into a collection $D(i)$ of nonintersecting pairs of adjacent intervals,

(2) no interval in $C(i+1)$ has an end point which is also the end point of an interval in $C(q)$, $q = 1, 2, \cdots, i$,

(3) if $(x, y) \in C(i)$, then $G(x, y) < 1/3$, and

(4) $\Sigma_{C(i)} |G| > i$.

Let $C = \bigcup_1^\infty D(i)$, and let F be the function on $[a, b]$ such that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geqq G(r, s)$, then

(a) $F(u, v) = -2$ if $G(u, v) < 0$,

(b) $F(u, v) = 2$ if $G(u, v) \geqq 0$,

(c) $F(r, x) = -1$ if $r = v$ and $r < x$, and

(d) $F(x, s) = -1$ if $s = u$ and $x < s$,

and $F(x, y) = 0$ otherwise. Thus, $F \in OP^\circ$ on $[a, b]$. However,

$$[1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] \geqq 1 + |G(u, v)|/3 .$$

Hence, since G is bounded and $\{\Sigma_{C(i)} |G|\}_1^\infty$ is unbounded, $FG \notin OP^\circ$. This is a contradiction, and therefore, $G \in OB^\circ$ on $[a, b]$.

LEMMA 5.2. *If G is a function such that*

(1) *G is almost bounded below by 1/10 on $[a, b]$, and*

(2) *if $F \in OP^\circ$ on $[a, b]$, then $FG \in OP^\circ$ on $[a, b]$,*

then $G - 1 \in OB^\circ$ on $[a, b]$.

Proof. Suppose $G - 1 \notin OB^\circ$ on $[a, b]$. It follows from Theorem 4 that G is bounded on $[a, b]$. There exists a set $\{C(i)\}_1^\infty$ satisfying conditions (1) and (2) in Lemma 5.1 plus the additional conditions

(3) if $(x, y) \in C(i)$, then $G(x, y) > 1/10$, and

(4) $\Sigma_{C(i)} |G - 1| > i$.

Let $C = \bigcup_1^\infty D(i)$, where $D(i)$ is defined as in Lemma 5.1. Note that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geqq G(r, s)$, then either

(5) $G(u, v) \geqq 1$ and $|1 - G(u, v)| \geqq |1 - G(r, s)|$, or

(6) $G(r, s) < 1$ and either $G(u, v) = G(r, s)$ or

$$|1 - G(u, v)| < |1 - G(r, s)|.$$

Let F be the function on $[a, b]$ such that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geqq G(r, s)$, then

(a) $F(u, v) = -2$ and $F(r, s) = 0$ if (5) is true,

(b) $F(u, v) = 1$ and $F(r, x) = -1/2$ if (6) is true, $r = v$ and $r < x$, and

(c) $F(u, v) = 1$ and $F(x, s) = -1/2$ if (6) is true, $s = u$ and $x < s$,

and $F(x, y) = 0$ otherwise. Thus, $F \in OP^\circ$ on $[a, b]$. Observe that if (5) is true, then

$$[1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] = -\{1 + 2[G(u, v) - 1]\} ,$$

and if (6) is true, then

$$\begin{aligned}[1 + F(u, v)G(u, v)]&[1 + F(r, s)G(r, s)] \\ &\geqq [1 + G(r, s)][1 - G(r, s)/2] \\ &> 1 + [1/20][1 - G(r, s)] .\end{aligned}$$

Hence, since G is bounded and $\{\Sigma_{C(i)} | G - 1 |\}_1^\infty$ is unbounded, $FG \notin OP^\circ$. This is a contradiction, and therefore, $G - 1 \in OB^\circ$ on $[a, b]$.

THEOREM 5. *If G is a function, then the following are equivalent:*

(1) *$G \in OB^*$ on $[a, b]$, and*

(2) *if $F \in OP^\circ$ on $[a, b]$, then $FG \in OP^\circ$ on $[a, b]$.*

Proof $(2 \rightarrow 1)$. If $a \leqq \alpha < b$, then there exists a number β such that $\alpha < \beta \leqq b$ and either $G \in OB^\circ$ on $[\alpha, \beta]$ or $G - 1 \in OB^\circ$ on $[\alpha, \beta]$. If this is false and $a \leqq \alpha < \beta < b$, then it follows from Lemmas 5.1 and 5.2 that G is neither almost bounded above by 1/3 nor almost bounded below by 1/10 on $[\alpha, \beta]$; hence, there exist sequences $\{s_p\}_1^\infty$ and $\{r_p\}_1^\infty$ such that

(1) s_p and r_p are subintervals of $[a, b]$ with a common end point,

(2) s_p precedes r_p and r_{p+1} precedes s_p, and

(3) $G(s_p) < 1/10$ and $G(r_p) \geqq 1/10$.

Let $H = \{s_p\}_1^\infty \cup \{r_p\}_1^\infty$, and let F be the function on $[a, b]$ such that

(1) $F(x, y) = -1$ if there exists an interval $(z, y) \in H$ such that $x < y$ and $G(z, y) < 1/10$,

(2) $F(x, y) = 2$ if $(x, y) \in H$ and $G(x, y) \geqq 1/10$, and

(3) $F(x, y) = 0$ otherwise.

Thus, $F \in OP^\circ$ on $[a, b]$. However, it follows that $FG \notin OP^\circ$ on $[a, b]$ since

$$[1 + F(s_p)G(s_p)][1 + F(r_p)G(r_p)] > (.9)(1.2) = 1.08 .$$

Similarly, if $a < \beta \leqq b$, then there exists a number α such that $a \leqq \alpha < \beta$ and either $G \in OB^\circ$ on $[\alpha, \beta]$ or $G - 1 \in OB^\circ$ on $[\alpha, \beta]$. It now follows that $G \in OB^*$ on $[a, b]$ by using the covering theorem.

Proof $(1 \to 2)$. Since $OB^\circ \subseteqq OP^\circ$, if $G \in OB^\circ$ and $F \in OP^\circ$ on $[x, y]$, then $FG \in OP^\circ$ on $[x, y]$. Note that

$$1 + FG = 1 + F + F(G - 1) .$$

Thus, it follows from Theorem 1 that if $G - 1 \in OB^\circ$ and $F \in OP^\circ$ on $[x, y]$, then $FG \in OP^\circ$ on $[x, y]$. Therefore, (1) must imply (2).

COROLLARY 5.1. *If G is a function, then the following are equivalent*:

(1) $G \in OP^\circ$ *on* $[a, b]$, *and*

(2) *if* $F \in OB^*$ *on* $[a, b]$, *then* $FG \in OP^\circ$ *on* $[a, b]$.

Indication of proof. It follows that (1) implies (2) by using Theorem 5 and that (2) implies (1) by considering the function $F \equiv 1$.

LEMMA 6.1. *If G is a bounded function such that*

(1) *G is almost bounded above by* $1/3$ *on* $[a, b]$, *and*

(2) *if* $F \in OQ^\circ$ *and is bounded on* $[a, b]$ *and* $1 + FG$ *is bounded away from zero, then* $FG \in OQ^\circ$ on $[a, b]$,

then $G \in OB^\circ$ *on* $[a, b]$.

Proof. Suppose $G \notin OB^\circ$ on $[a, b]$. There exist a subdivision D of $[a, b]$ and a positive integer m such that if J is a refinement of D and $u \in J(I)$ then $|G(u)|/m < 1/2$. Let H be the set such that $u \in H$ only if there exists a refinement J of D such that $u \in J(I)$, and let F be the function such that

(1) $F(u) = -2$ if $u \in H$ and $0 \leqq G(u) \leqq 1/3$,

(2) $F(u) = 1/m$ if $u \in H$ and $G(u) < 0$, and

(3) $F(x, y) = 0$ otherwise.

Since $F \in OQ^\circ$ and $1 + FG$ is bounded away from zero, $FG \in OQ^\circ$. However, it follows from Lemma 3.1 that $FG \notin OQ^\circ$. This is a contradiction, and therefore, $G \in OB^\circ$.

LEMMA 6.2. *If G is a bounded function such that*

(1) *G is almost bounded below by* $1/10$ *on* $[a, b]$, *and*

(2) *if* $F \in OQ^\circ$ *and is bounded on* $[a, b]$ *and* $1 + FG$ *is bounded away from zero, then* $FG \in OQ^\circ$ *on* $[a, b]$,

then $G - 1 \in OB^\circ$ *on* $[a, b]$.

Proof. There exist a subdivision D of $[a, b]$ and a number B such that if J is a refinement of D and $u \in J(I)$ then $|G(u)| < B$. Let H be the set such that $u \in H$ only if there exists a refinement J of D such that $u \in J(I)$. Let H_1 and H_2 be the subsets of H such that $u \in H_1$ only if $G(u) \leqq 1$ and $u \in H_2$ only if $G(u) > 1$. For $i = 1, 2$, let $G_i(x, y) = G(x, y)$ if $(x, y) \in H_i$ and $G_i(x, y) = 0$ if $(x, y) \notin H_i$.

Suppose $G_1 - 1 \notin OB^\circ$ on $[a, b]$. Let F be the function such that

(1) $F(u) = -2$ if $u \in H_1$ and $G(u) < 5/12$ or $7/12 < G(u) \leqq 1$,

(2) $F(u) = -3$ if $u \in H_1$ and $5/12 \leqq G(u) \leqq 7/12$, and

(3) $F(x, y) = 0$ otherwise.

Since $F \in OQ^\circ$ and $1 + FG$ is bounded away from zero, $FG \in OQ^\circ$. However, it follows from Lemma 3.1 that $FG \notin OQ^\circ$. This is a contradiction, and therefore, $G_1 - 1 \in OB^\circ$.

Suppose $G_2 - 1 \notin OB^\circ$ on $[a, b]$. There exist a set $\{C(i)\}_1^\infty$ and an integer $m > 1$ such that

(1) $C(i)$ is a finite set of nonoverlapping subintervals of $[a, b]$ which can be grouped into a collection $D(i)$ of nonintersecting pairs $\{(u, v), (r, s)\}$ of adjacent intervals such that either $G(u, v) > 1$ or $G(r, s) > 1$,

(2) no interval in $C(i + 1)$ has an end point which is also the end point of an interval in $C(q)$, $q = 1, 2, \cdots, i$,

(3) if $(x, y) \in C(i)$ then $G(x, y) > 1/10$ and $G(x, y)/m < 1/2$, and

(4) $\Sigma_{C(i)} |G_2 - 1| > i$.

Let $C = \bigcup_1^\infty D(i)$, and let F be the function such that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geqq G(r, s)$ then $F(u, v) = -1/m$, $F(r, x) = 1/(m - 1)$ if $r = v$ and $F(x, s) = 1/(m - 1)$ if $s = u$, and $F(x, y) = 0$ otherwise. Since $F \in OQ^\circ$ and $1 + FG$ is bounded away from zero, $FG \in OQ^\circ$. However, if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geqq G(r, s)$, then

$$\begin{aligned} 0 &< [1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] \\ &\leqq [1 - G(u, v)/m][1 + G(u, v)/(m - 1)] \\ &< 1 + [1 - G(u, v)]/m(m - 1) . \end{aligned}$$

It follows from Lemma 3.1 that $FG \notin OQ^\circ$. This is a contradiction, and therefore, $G_2 - 1 \in OB^\circ$.

Thus, since $G_i - 1 \in OB^\circ$ on $[a, b]$ for $i = 1, 2$, it follows that $G - 1 \in OB^\circ$ on $[a, b]$.

THEOREM 6. *If G is a bounded function, then the following are equivalent*:

(1) *$G \in OB^*$ on $[a, b]$, and*

(2) *if $F \in OQ^\circ$ and is bounded on $[a, b]$ and $1 + FG$ is bounded away from zero, then $FG \in OQ^\circ$ on $[a, b]$.*

Proof ($2 \rightarrow 1$). If $a \leqq \alpha < b$, then there exists a number β such that $\alpha < \beta \leqq b$ and either $G \in OB^\circ$ on $[\alpha, \beta]$ or $G - 1 \in OB^\circ$ on $[\alpha, \beta]$. If this is false, then it follows from Lemmas 6.1 and 6.2 that there exist sequences $\{s_p\}_1^\infty$ and $\{r_p\}_1^\infty$ and a set H defined as in Theorem 5. Let F be a function on $[a, b]$ such that if (u, v) and (v, s) are intervals in H such that $G(u, v) \leqq 1/10$ and $G(v, s) \geqq 1/10$, then

(1) $1 + F(u, v)G(u, v) = 1/2$ and $F(v, s) = 0$ if $G(u, v) < -1/10$,

(2) $F(x, v) = 1$, $-1/2 \leqq F(v, s) < 0$ and $1/2 \leqq 1 + F(v, s)G(v, s) \leqq .95$ if $-1/10 \leqq G(u, v) \leqq 0$, and

(3) $F(x, v) = -3$, $-1/2 \leqq F(v, s) < 0$ and $1/2 \leqq 1 + F(v, s)G(v, s) \leqq .95$ if $0 < G(u, v) < 1/10$,

and $F(x, y) = 0$ otherwise. Since F is a bounded function in OQ° such that $1 + FG$ is bounded away from zero, $FG \in OQ^\circ$. However,

$$|[1 + F(s_p)G(s_p)][1 + F(r_p)G(r_p)]| \leqq .95 .$$

Hence, $FG \notin OQ^\circ$. Similarly, if $a < \beta \leqq b$, then there exists a number α such that $a \leqq \alpha < \beta$ and either $G \in OB^\circ$ on $[\alpha, \beta]$ or $G - 1 \in OB^\circ$ on $[\alpha, \beta]$. It now follows that $G \in OB^*$ on $[a, b]$ by using the covering theorem.

Proof $(1 \rightarrow 2)$. This follows from Theorem 3 by a procedure similar to that used in Theorem 5.

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