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THE MULTIPLIER ALGEBRA OF A CONVOLUTION MEASURE ALGEBRA

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In this paper the structure theory of convolution measure algebras due to J. L. Taylor is used in studying the multiplier algebra $M(A)$ of a commutative semi-simple convolution measure algebra A. A criterion is given for the embeddability of $M(A)$ in the measure algebra $M(S)$ on the structure semigroup S of A, and the relationship between the structure semigroups of A and $M(A)$ is investigated in case $M(A)$ is also a convolution measure algebra and S has an identity.

1. Introduction. A convolution measure algebra A is a complex L-space with a multiplication which gives A the structure of a Banach algebra and satisfies certain additional requirements. For precise definitions and the basic theory of convolution measure algebras we refer to J. L. Taylor's paper [11]. A central role in Taylor's theory is played by the structure semigroup S of a commutative convolution measure algebra A. The maximal regular ideal space of A may be identified with the set of semicharacters of the compact commutative topological semigroup S, and some properties of A are reflected in those of S.

For any (complex) commutative Banach algebra A, let $\Delta(A)$ denote the spectrum of A, that is, the space of nonzero multiplicative linear functionals on A, equipped as usual with the relative weak* topology. If A is in addition semisimple, then we denote by A^m the space of all complex-valued functions on $\Delta(A)$ that keep the space $\hat{A}$ of the Gelfand transforms $\hat{x}$ of the elements x of A invariant by pointwise multiplication, i.e., $A^m = \{f: \Delta(A) \to C \mid f\hat{x} \in \hat{A} \text{ for all } x \in A\}$. It can be easily shown that each $f \in A^m$ determines a unique bounded linear operator $T_f: A \to A$ satisfying $\widehat{T_f x} = f\hat{x}$, $x \in A$. Then $M(A) = \{T_f \mid f \in A^m\}$ is a Banach algebra under the uniform operator norm, called the *multiplier algebra* of A. For the general theory of multiplier algebras one may consult e.g. Larsen's book [5].

In this paper we study the multiplier algebra of a commutative semi-simple convolution measure algebra A. J. L. Taylor has shown in [11] that A may be naturally embedded in the convolution algebra $M(S)$ of finite regular Borel measures on the structure semigroup S. In § 3 we show that $M(A)$ can be isometrically realized as a subalgebra of $M(S)$ containing the image of A if and only if S has an identity. As is to be expected, the measures then corresponding to isometric onto multipliers have one point support in S. Section 4 gives conditions for $M(A)$ to be a convolution measure algebra, too, and § 5 concentrates on describing the relationship that exists between S and the structure semigroup of $M(A)$ provided $M(A)$ is a convolution measure algebra and S has an identity. For related results in a somewhat different situation, see [13].

For any compact Hausdorff space S, $C(S)$ will denote the Banach space of continuous complex-valued functions on S with the supremum norm, and $M(S)$ is the conjugate space of $C(S)$. Of course, $M(S)$ may be interpreted as the space of finite regular Borel measures on S, and if S is also a topological semigroup, $M(S)$ is a Banach algebra under the convolution product

$$\mu*\nu(f) = \int_S \int_S f(xy) d\mu(x) d\nu(y) .$$

2. Taylor's structure semigroup of a commutative convolution measure algebra. Preliminarily to our discussion of the multiplier algebra we give in this section the structure semigroup a description which differs slightly from Taylor's original construction. In special cases an essentially similar method has been used e.g. by Rennison in [8] and Ramirez in [7]. See also [6] and [13].

The conjugate space A' of any complex L-space A is a commutative C^*-algebra with identity. The corollary in [11, p. 157] says that if A is a commutative convolution measure algebra, then $\Delta(A) \cup \{0\}$ is a self-adjoint multiplicative subsemigroup of A' containing the identity, so that the norm closed linear span P of $\Delta(A)$ in A' is a C^*-algebra with identity. A *semicharacter* on a topological semigroup is a non-zero continuous homomorphism into the multiplicative semigroup of complex numbers z with $|z| \leqq 1$.

THEOREM 2.1. *Let A be a commutative convolution measure algebra and P as above. For any $F, G \in P'$ there is a unique element, denoted FG, of P' such that $FG(\alpha) = F(\alpha)G(\alpha)$ for all $\alpha \in \Delta(A)$. The map $(F, G) \mapsto FG$ is a commutative Banach algebra product in P'. The spectrum $\Delta(P)$ of P is a multiplicative subsemigroup of P'. With the relative weak* topology $\Delta(P)$ is a compact topological semigroup, and the semicharacters of $\Delta(P)$ are precisely the Gelfand transforms of the elements of $\Delta(A)$. The structure semigroup S of A in the sense of Taylor* [11] *is topologically isomorphic to $\Delta(P)$.*

Proof. The product in P' that we are referring to is discussed in [1, p. 816] and [13, pp. 168–169]. In particular, since $FG(\alpha\beta) = F(\alpha\beta)G(\alpha\beta)$ for all $\alpha, \beta \in \Delta(A)$, $F, G \in P'$, even if $\alpha\beta = 0$, the proof of Theorem 2.3 in [13] is valid also in our present situation where, in general, merely $\Delta(A) \cup \{0\}$ is a multiplicative subsemigroup of A'. Similarly, Theorem 2.4 in [13] is applicable, for the semi-simplicity of A is nowhere needed in its proof, and $\Delta(A)$ (rather than $\Delta(A) \cup \{0\}$) is assumed to be closed with respect to multiplication only to allow one to appeal to the above mentioned Theorem 2.3. From the proof of Theorem 2.2 in [11] it is clear that there is a homeomorphism φ from the structure semigroup S of A onto $\Delta(P)$ such that its natural dual map from $C(\Delta(P))$ onto $C(S)$ puts the sets of semicharacters on S and $\Delta(P)$ in a bijective correspondence. As in the proof of Theorem 6.5 in [7] it is seen that φ is also a semigroup isomorphism.

From now on we call $\Delta(P)$ with the product mentioned in the above theorem the *structure semigroup* of A and use the notation $S = \Delta(P)$.

THEOREM 2.2. *Let A and P be as in Theorem 2.1. If P' is given the product referred to in that theorem, then the isometric embedding $F \mapsto \mu_F$ from P' onto $M(S) = C(S)'$ defined by $\langle f, \mu_F \rangle = \langle f, F \rangle$ for $F \in P'$, $f \in P = C(S)$, is an algebra isomorphism.*

Proof. Suppose $F, G \in P'$. By the definition of the convolution $\mu_F * \mu_G$ we have for any $\alpha \in \Delta(A)$,

$$\begin{aligned} \langle \hat{\alpha}, \mu_F * \mu_G \rangle &= \int_S \int_S \hat{\alpha}(xy) d\mu_F(x) d\mu_G(y) = \int_S \hat{\alpha}(x) d\mu_F(x) \int_S \hat{\alpha}(y) d\mu_G \\ &= \langle \alpha, F \rangle \langle \alpha, G \rangle = \langle \alpha, FG \rangle = \langle \hat{\alpha}, \mu_{FG} \rangle . \end{aligned}$$

Since the functions $\hat{\alpha}$, $\alpha \in \Delta(A)$, generate the Banach space $C(S)$, the equality $\langle h, \mu_F * \mu_G \rangle = \langle h, \mu_{FG} \rangle$ is valid for all $h \in C(S)$, i.e., $\mu_F * \mu_G = \mu_{FG}$.

3. Representing the multipliers as measures on the structure semigroup. Throughout the rest of the paper we assume that A is a commutative, *semi-simple* convolution measure algebra with structure semigroup $S = \Delta(P)$, where P is always the closed linear span of $\Delta(A)$ in A'. The set of semicharacters on S is denoted by $\hat{S}$. We give P' the Banach algebra product mentioned in Theorem 2.1.

LEMMA 3.1. *The natural embedding $\theta: A \rightarrow P'$ defined by $\langle f, \theta x \rangle = \langle x, f \rangle$, $x \in A$, $f \in P$, is an isometric and bipositive* (i.e., $\theta x \geqq 0$ *if and only if* $x \geqq 0$) *algebra homomorphism.*

Proof. From the definition of the product in P' it is clear that θ is a homomorphism. Theorem 2.3 in [11] along with the corollary in [11, p. 154] shows that it is isometric and bipositive. Alternatively, θ is isometric by virtue of the Kaplansky density theorem [10, p. 22], and bipositive by Propositions 1.5.1 and 1.5.2 in [10, p. 9], since P

contains the identity of A'.

We usually identify P' and $M(S)$ as ordered Banach spaces and Banach algebras in accordance with Theorem 2.2. Following Taylor [11], we use the notation $\theta(A) = A_S \subset M(S)$. It follows e.g. from [11, Theorem 2.3] and the corollary in [11, p. 154] that A_S is an L-subspace [11, p. 151] of the complex L-space $M(S)$.

LEMMA 3.2. *The convolution product in $M(S)$ is separately weak* continuous.*

Proof. Suppose $\nu \in M(S)$ and $f \in C(S)$. It is a simple matter to show that the function φ, $\varphi(y) = \int_S f(xy)d\nu(x)$, is continuous on S. (A much more general result may be proved using Grothendieck's weak compactness theorem, see [3, p. 205].) Since we have $|\langle f, \nu*\mu\rangle| = \left|\int_S\int_S f(xy)d\nu(x)d\mu(y)\right| = |\langle\varphi, \mu\rangle|$, the mapping $\mu \mapsto \nu*\mu$ is weak* continuous at zero, hence everywhere.

LEMMA 3.3. *Suppose that S has an identity and $\mu \in M(S)$. Then $\mu \geqq 0$ if and only if $\mu*\nu \geqq 0$ for all $\nu \geqq 0$ in A_S.*

Proof. Clearly the latter condition is necessary. Suppose now that $\mu*\nu \geqq 0$ for all $\nu \geqq 0$ in A_S. Choose a basis $\mathscr{U}$ of compact neighborhoods of the identity u of S, directed in the natural order opposite to inclusion. Each $\nu \in A_S$ is a linear combination of non-negative elements of A_S and if $\lambda \in M(S)$, $0 \leqq \lambda \leqq \nu \in A_S$, then $\lambda \in A_S$ since A_S is an L-subspace of $M(S)$. Furthermore, since A_S separates $P = C(S)$, A_S is a weak* dense subspace of $M(S)$. It follows easily that for each $U \in \mathscr{U}$ there exists a positive measure $\mu_U \in A_S$, $\|\mu_S\| = 1$, with support contained in U. The net $(\mu_U)_{U\in\mathscr{U}}$ converges to the Dirac measure δ_u in the weak* topology. By assumption, $\mu*\mu_U \geqq 0$ for all $U \in \mathscr{U}$, and since the positive cone in $M(S)$ is weak* closed and the convolution is separately weak* continuous (Lemma 3.2), it follows that $\mu = \mu*\delta_u = \lim_U \mu*\mu_U \geqq 0$.

We regard the multiplier algebra $M(A)$ as an ordered Banach space with positive cone $\{T \in M(A) \mid Tx \geqq 0 \text{ for all } x \geqq 0 \text{ in } A\}$.

THEOREM 3.1. *If S has an identity, then there exists a bipositive, isometric algebra isomorphism from $M(A)$ onto the subalgebra $B = \{\mu \in M(S) \mid \mu*\nu \in A_S \text{ for all } \nu \in A_S\}$ of $M(S)$. Conversely, if there exists an isometric algebra isomorphism ψ from $M(A)$ onto a subalgebra of $M(S)$ containing A_S, then S has an identity, and for any isometric and surjective $T \in M(A)$ we have $\psi(T) = c\delta_x$, where $c \in \boldsymbol{C}$, $|c| = 1$ and δ_x is the Dirac measure corresponding to some $x \in S$.*

Proof. Suppose that S has an identity u. The net $(\mu_U)_{U\in\mathscr{U}}$ constructed in the proof of Lemma 3.3 converges to the Dirac measure δ_u in the weak* topology of $M(S)$. In particular, $\lim_U \langle \mu_U, \gamma\rangle = 1$ for each $\gamma \in \Delta(A) = \hat{S}$. If we denote by $T_f \in M(A)$ the operator corresponding to the function $f \in A^m$ (see the introduction), an argument given in [1, p. 817] shows that $|\sum_{k=1}^n a_k f(\gamma_k)| \leqq \| T_f \| \| \sum_{k=1}^n a_k \gamma_k \|$ for all $\gamma_k \in \Delta(A)$, $a_k \in \boldsymbol{C}$, $k = 1, \cdots, n$. It follows that f can be extended as a continuous linear functional $\boldsymbol{f}$ to the whole of P with norm $\|\boldsymbol{f}\| \leqq \|T_f\|$. Since the embedding $\theta\colon A \to P'$ is isometric (Lemma 3.1), we have, using the definition of the product in P', $\|\boldsymbol{f}\| \geqq \sup\{\|\boldsymbol{f}\theta(x)\| \mid x \in A,\ \|\theta(x)\| \leqq 1\} = \sup\{\|\boldsymbol{f}\theta(x)\| \mid x \in A,\ \|x\| \leqq 1\} = \sup_{\substack{\|x\|\leqq 1\\ x\in A}} \|T_f x\| = \|T_f\|$. Thus $\|\boldsymbol{f}\| = \|T_f\|$. From the definition of the product in P' it is obvious that the above embedding of $M(A)$ in P' is an algebra homomorphism, so that it may be interpreted as an isometric algebra homomorphism $\pi\colon M(A) \to M(S)$ (Theorem 2.2). Since the embedding of A in $M(S)$ is bipositive (Lemma 3.1), it is clear from Lemma 3.3 that π is bipositive. Denote $\pi(M(A)) = B \subset M(S)$. For functions in $A^m(\supset \hat{A})$, pointwise multiplication corresponds to the convolution of the respective measures on S (see the proof of Theorem 2.2). Therefore, A_S is an ideal in B. Also, if $\mu \in M(S)$, and $\mu * \nu \in A_S$ for all $\nu \in A_S$, then the function $f_\mu\colon \Delta(A) \to \boldsymbol{C}$ obtained by restricting μ to $\hat{S} = \Delta(A)$ belongs to A^m. The first part of the theorem is thus proved. To prove the converse assertion we note that $M(A)$ has always an identity. The hypothesis then implies that a weak* dense subalgebra of $M(S)$ has an identity η. It follows from Lemma 3.2 that η is also an identity for $M(S)$. But it is well known that the identity of any Banach algebra is an extreme point in its unit ball (see e.g. [10, p. 13]). Hence (see [2, p. 441]) we have $\eta = c\delta_u$ for some $u \in S$ and $c \in \boldsymbol{C}$, $|c| = 1$. In fact $c = 1$, since $c\delta_u = c\delta_u * c\delta_u = c^2\delta_{u^2}$, so that $u = u^2$ and $c = c^2$. Now, u is an identity for S, since $\delta_{ux} = \delta_u * \delta_x = \delta_x$ for all $x \in S$. For the last statement, it is enough to show that $\psi(T)$ is an extreme point of the unit ball of $M(S)$ [2, p. 441]. If $0 \leqq \lambda \leqq 1$ and $\mu_1, \mu_2 \in M(S)$ are such that $\psi(T) = \lambda\mu_1 + (1-\lambda)\mu_2$ and $\|\mu_1\| \leqq 1$, $\|\mu_2\| \leqq 1$, we have for the identity η of $M(S)$, since also $T^{-1} \in M(A)$ [5, p. 15],

$$\eta = \psi(T^{-1}) * \psi(T) = \lambda\psi(T^{-1}) * \mu_1 + (1-\lambda)\psi(T^{-1}) * \mu_2\,,$$

where $\|\psi(T^{-1}) * \mu_1\| \leqq 1$ and $\|\psi(T^{-1}) * \mu_2\| \leqq 1$. Since η is an extreme point of the unit ball of $M(S)$, we have $\lambda = 0$ or $\lambda = 1$. Therefore, $\psi(T)$ is also an extreme point of the unit ball of $M(S)$.

NOTE. From the proof of the above theorem it is clear that S has an identity if and only if A has a weak bounded approximate identity [1, p. 817] of norm one. (Compare [11, Theorem 3.1].)

4. **$M(A)$ as a convolution measure algebra.** If S has an identity, then $M(A)$ may be embedded in $M(S)$ in accordance with Theorem 3.1. Unfortunately, the nature of the image $B \subset M(S)$ of $M(A)$ is not sufficiently clear on the basis of that result. For example, we should like to conclude that B is a so-called L-subalgebra of $M(S)$, which turns out to be equivalent to saying that $M(A)$ with its natural order is a convolution measure algebra. The next theorem gives two other necessary and sufficient conditions for this to be case.

We assume henceforth that S *has an identity* and let $\pi: M(A) \to M(S)$ be the bipositive, isometric homomorphism constructed in the proof of Theorem 3.1, and denote as before $B = \pi(M(A)) = \{\mu \in M(S) \mid \mu * \nu \in A_S$ for all $\nu \in A_S\}$. Since S has an identity, $\Delta(A)$ (and not merely $\Delta(A) \cup \{0\}$) is a multiplicative subsemigroup of A', so that it makes sense to talk about translations of functions on $\Delta(A)$. A set $\mathscr{F}$ of functions $f: \Delta(A) \to \boldsymbol{C}$ is *translation invariant*, if $f \in \mathscr{F}$ implies $f^\alpha \in \mathscr{F}$ for all $\alpha \in \Delta(A)$, where $f^\alpha(\beta) = f(\alpha\beta)$ for $\alpha, \beta \in \Delta(A)$.

A closed subalgebra N of the convolution measure algebra $M(S)$ is an *L-subalgebra* of $M(S)$, if for any $\mu \in N$ its total variation $|\mu|$ belongs to N, and if $\nu \in N$ whenever $\mu \in N$ and ν is absolutely continuous with respect to $|\mu|$ (denoted $\nu \ll |\mu|$) [12, p. 257]. This definition is easily seen to be equivalent to requiring that N be a subalgebra and an L-subspace of $M(S)$ in the sense of [11].

THEOREM 4.1. *The following conditions are equivalent*:

(i) *$M(A)$ is a convolution measure algebra* (*in the order defined before Theorem* 3.1),

(ii) *B is an L-subalgebra of $M(S)$*,

(iii) *A^m is a translation invariant set of functions on $\Delta(A)$*,

(iv) *for any $\mu \in B$, $|\mu|$ also belongs to B.*

Proof. We shall establish the following chain of implications: (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). If (ii) holds and $\mu \in B, f \in C(S)$, then the measure $f\mu$ (i.e., the functional $g \mapsto \mu(fg)$ on $C(S)$) belongs to B. But if $f \in A^m$ and $\mu_f = \pi(T_f)$, then we have $f^\alpha(\beta) = \mu_f(\alpha\beta) = \alpha\mu_f(\beta)$ for all $\alpha, \beta \in \Delta(A) = \hat{S}$, so that $f^\alpha \in A^m$, since $\alpha\mu_f \in B$ and A_S is an ideal in B. Thus (ii) implies (iii). Next, assume (iii) and choose any $\mu \in B$. Then the function $f_\mu: \Delta(A) \to \boldsymbol{C}$ defined by $f_\mu(\alpha) = \mu(\alpha)$ for $\alpha \in \Delta(A) = \hat{S}$ belongs to A^m. By assumption, $(f_\mu)^\alpha \in A^m$ for any $\alpha \in \Delta(A)$. But the measure in B corresponding to $(f_\mu)^\alpha$ when α is regarded as a semicharacter of S, is $\alpha\mu$. As $\hat{S}$ generates the Banach space $C(S)$ and the mapping $f \mapsto f\mu$ is continuous from $C(S)$ to $M(S)$, which contains B as a closed subspace, we have $f\mu \in B$ for all $f \in C(S)$. Furthermore, $C(S)$ is dense in $L^1(S, |\mu|)$ [4, p. 140] so that there is a sequence of functions in $C(S)$ converging to $\bar{g}$ in $L^1(S, |\mu|)$, where g is a $|\mu|$-measurable function with $|g(x)| = 1$ $|\mu|$-a.e. and such that $\mu = g|\mu|$ [4, p. 171]. By virtue of the continuity of the mapping $f \mapsto f\mu$ from $L^1(S, |\mu|)$ to $M(S)$ and the fact that $f\mu \in B$ for all $f \in C(S)$, it follows that $|\mu| = \bar{g}\mu \in B$, i.e., (iv) holds. Assume now (iv) and that $\mu \in M(S)$ is absolutely continuous with respect to some $\lambda \geqq 0$ in B. Then we have also $\mu_j \ll \lambda$, $j = 1, \cdots, 4$, in the Jordan decomposition $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where μ_1 and μ_2 (resp. μ_3 and μ_4) are mutually singular nonnegative measures. If $\nu \geqq 0$ is in A_S, we have $\nu * \mu_j \ll \nu * \lambda$. This has been proved in a somewhat more general setting by Pym in [6, p. 630]. Since A_S is an L-subspace of $M(S)$, hence an L-subalgebra in the sense of [12], we have $\nu * \mu \in A_S$. It follows that in fact $\nu * \mu \in A_S$ for an arbitrary $\nu \in A_S$, so that $\mu \in B$. Thus (ii) holds. Since $M(A)$ is isometrically algebra and order isomorphic to B, and any L-subalgebra of $M(S)$ is a convolution measure algebra (see [11, p. 151 and Definition 2.1]), (ii) implies (i) at once. Finally, if $M(A)$ is a convolution measure algebra (hence a complex L-space), (iv) holds by virtue of Corollary 1.6 and Proposition 1.8 in [9].

On the basis of the above theorem sufficient conditions (assuming that S has an identity) can be given to guarantee that $M(A)$ is a convolution measure algebra. Since A_S is an L-subalgebra of $M(S)$, an argument used in the proof of the above theorem shows that $\hat{A}$ is translation invariant on $\Delta(A)$. If we assume for example that A *is regular and has a bounded approximate identity consisting of elements with Gelfand transforms of compact support*, then the theorem in [1, p. 819] shows that A^m consists of those functions on $\Delta(A)$ which belong locally to $\hat{A}$ and may be extended to continuous linear functionals on P. Then Theorem 4.2 in [13] shows that A^m is translation invariant. Another case where the translation invariance of A^m follows immediately from that of $\hat{A}$ arises, when S is a multiplicative group, for then we have $(f^\alpha \hat{x})(\beta) = f(\alpha\beta)\hat{x}^{\alpha^{-1}}(\alpha\beta) = \hat{y}(\beta)$ for some $y \in A$, if $f \in A^m$ and $x \in A$. For a discussion of this kind of a situation, see [12].

5. The structure semigroups of A and $M(A)$. We retain the general hypotheses and notational conventions made in §§3 and 4. In particular S *has an identity*. Let us make the additional assumption that $M(A)$ *is a convolution measure algebra* (in the order defined before Theorem 3.1). When A and $M(A)$ are realized as subalgebras of $M(S)$ (§3), it is seen that the embedding $j: A \to M(A)$ defined by $j(x) = T_{\hat{x}}$ is isometric and bipositive. It is readily seen to be an L-homomorphism [11, p. 152], since A_S is an L-subspace of $M(S)$. We let Q denote the closed linear span of $\Delta(M(A))$ in $M(A)'$. Then $T =$

$\Delta(Q)$ with the usual topology and product is the structure semigroup of $M(A)$. Before stating Theorem 5.1, which relates S and T to each other, we prove an auxiliary result.

LEMMA 5.1. *The mapping* $\Phi: P \rightarrow M(A)'$ *defined by* $\langle \Phi f, L\rangle = \langle \pi(L), f\rangle$ *for* $L \in M(A)$, $f \in P = C(S)$, *is an isometric* C^**-algebra homomorphism which maps the identity of* P *to that of* $M(A)'$, *and we have*

$$(1) \qquad j^* \circ \Phi(f) = f\,, \qquad f \in P\,,$$

for the transpose $j^*: M(A)' \rightarrow A'$ *of* j. *Furthermore,* $\Phi(P) \subset Q$.

Proof. Equation (1) is immediate. Since $j: A \rightarrow M(A)$ is an L-homomorphism, $j^*: M(A)' \rightarrow A'$ is a C^*-algebra homomorphism which preserves the identity [11, p. 153]. Therefore,

$$(2) \qquad j^*(\Phi\alpha\Phi\beta) = \alpha\beta \quad \text{for} \quad \alpha, \beta \in \Delta(A) = \hat{S}\,.$$

As S has an identity, $\alpha\beta \neq 0$. A simple calculation shows that since π is a homomorphism, $\Phi\alpha$ and $\Phi\beta$ are multiplicative, so that by (2) their product is a multiplicative extension of $\alpha\beta$ to $M(A)$ (when $\alpha\beta$ is regarded as a functional on $j(A)$). Now, $\Phi(\alpha\beta)$ is also a multiplicative extension of $\alpha\beta$ to $M(A)$, and since there are only one of them [5, p. 24], we have $\Phi(\alpha\beta) = \Phi\alpha\Phi\beta$. A similar argument shows that $\Phi \mid \Delta(A)$ preserves involution. It follows that Φ is a C^*-algebra homomorphism. Since the identity e_1 of A' belongs to $\Delta(A)$ and the identity e_2 of $M(A)'$ to $\Delta(M(A))$, the uniqueness of the multiplicative extension again shows that $\Phi e_1 = e_2$. Since any C^*-algebra homomorphism is norm-decreasing [10, p. 5] it follows from (1) that Φ is isometric. The last statement is a consequence of the fact that $\Phi(\Delta(A)) \subset \Delta(M(A))$.

In the following theorem $\mathfrak{H}$ denotes the natural embedding of $M(A)$ in $M(T)$ [11, p. 158]. The identity map of a set D is denoted by id_D.

THEOREM 5.1. *There exist unique continuous semigroup homomorphisms* $\psi: S \rightarrow T$ *and* $\varphi: T \rightarrow S$ *such that*

(1) $\Phi f(t) = f \circ \varphi(t)$ *and* $\Psi g(s) = g \circ \psi(s)$

for all $t \in T$, $f \in C(S) = P$, $s \in S$, $g \in C(T) = Q$, *where* $\Psi = j^* \mid Q$ *and* Φ *is the map defined in Lemma* 5.1. *Furthermore,*

(a) $\varphi \circ \psi = id_S$ *and* $\Psi \circ \Phi = id_P$.

(b) $\psi(S)$ *is a closed ideal in* T.

(c) *For the identity* u *of* S *we have* $\psi \circ \varphi(t) = t\psi(u)$, $t \in T$.

(d) *If we denote* $M_S(T) = \{\mu \in M(T) \mid |\mu|(T \backslash \psi(S)) = 0\}$, *then* $M_S(T)$ *is an ideal in* $M(T)$ *and* $\Psi^*(M(S)) = M_S(T)$ *for the transpose* $\Psi^*: M(S) \rightarrow M(T)$ *of* Ψ.

(e) *The diagram below commutes, and all maps appearing in it are algebra homomorphisms.*

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "\theta"] & M(A) \arrow[d, "\mathfrak{H}"] \arrow[ddr, "\pi"] & \\
M(S) \arrow[r, "\Psi^*"] \arrow[drr, "id"'] & M(T) \arrow[dr, "\Phi^*"] & \\
& & M(S)
\end{tikzcd}$$

Proof. It is clear that (1) holds if and only if the maps $\psi\colon \Delta(P) \to \Delta(Q)$ and $\varphi\colon \Delta(Q) \to \Delta(P)$ are defined by setting $\langle \psi(s), g\rangle = \Psi g(s)$ and $\langle \varphi(t), f\rangle = \Phi f(t)$ for $s \in S$, $t \in T$, $g \in Q = C(T)$, and $f \in P = C(S)$. From the definition of the product in S and T it follows that ψ and φ, obviously continuous, are semigroup homomorphisms. For example, if $\gamma \in \Delta(M(A))$ and $x, y \in S$, then $\Psi\gamma \in \Delta(A)$ or $\Psi\gamma = 0$, and in both cases $\langle \psi(xy), \gamma\rangle = \langle xy, \Psi\gamma\rangle = \langle x, \Psi\gamma\rangle\langle y, \Psi\gamma\rangle = \langle \psi(x), \gamma\rangle\langle \psi(y), \gamma\rangle$, i.e., $\psi(xy) = \psi(x)\psi(y)$. The second formula in (a) is a consequence of Lemma 5.1, and the first formula follows from the second by a simple calculation. The commutativity of the square in (e) is seen as follows: $\langle g, \Psi^* \circ \theta(x)\rangle = \langle \Psi g, \theta x\rangle = \langle x, \Psi g\rangle = \langle j(x), g\rangle = \langle g, \mathfrak{H} \circ j(x)\rangle$ for $x \in A$, $g \in C(T) = Q$, so that $\Psi^* \circ \theta = \mathfrak{H} \circ j$. The lower triangle commutes because of (a). As to the upper triangle, note that if γ belongs to $\Delta(A)$, then $\Phi\gamma$ is its unique multiplicative extension to $M(A)$ (see the proof of Lemma 5.1). Therefore, $\langle \gamma, \Phi^* \circ \mathfrak{H}(L)\rangle = \langle \Phi\gamma, \mathfrak{H}(L)\rangle = f_L(\gamma) = \langle \gamma, \pi(L)\rangle$, where f_L is the function in A^m corresponding to $L \in M(A)$. Since $\Delta(A) = \hat{S}$ generates $C(S)$, the equation $\Phi^* \circ \mathfrak{H} = \pi$ follows. The second statement in (e) is also easily proved. Next we show that $\Psi^*(M(S)) = M_S(T)$. Since Ψ and Φ are norm-decreasing, Ψ^* and Φ^* are so, and (e) implies that Ψ^* is isometric. On the other hand, Ψ^* is continuous from $\sigma(M(S), C(S))$ to $\sigma(M(T), C(T))$. Therefore, using the weak* compactness of $B_r = \{\mu \in M(S) \mid \|\mu\| \leqq r\}$, we see that $\{\mu \in \Psi^*(M(S)) \mid \|\mu\| \leqq r\}$ is weak* compact, hence closed for each $r \geqq 0$. The Krein-Smulian theorem [2, p. 429] then shows that $\Psi^*(M(S))$ is weak* closed in $M(T)$. If S (resp. T) is considered naturally embedded in $M(S)$ (resp. $M(T)$), then $\Psi^* \mid S = \psi$, so that $\Psi^*(S) = \psi(S)$. The linear combinations of the Dirac measures are weak* dense in $M(S)$. Similarly, since $\sigma(M_S(T), C(T))$ and $\sigma(M_S(T), C(\psi(S)))$ coincide on $M_S(T)$, the linear span of $\psi(S) = \Psi^*(S)$ is $\sigma(M_S(T), C(T))$-dense in $M_S(T)$, which in turn is weak* closed in $M(T)$, as $\psi(S) \subset T$ is compact. From these remarks the equation $\Psi^*(M(S)) = M_S(T)$ follows by the weak* continuity of Ψ^*. From (e) and the fact that $j(A)$ is an ideal in $M(A)$ it follows that $\Psi^*(A_S)$ is an ideal in $\mathfrak{H}(M(A))$. Since A_S is weak* dense in $M(S)$ [11, p. 158], it follows from what was said above that the weak* closure of $\Psi^*(A_S)$ is $M_S(T) = \Psi^*(M(S))$. By virtue of the separate weak* continuity of the convolution in $M(T)$ (Lemma 3.2), M_S is therefore an ideal in $M(T)$, which contains $\mathfrak{H}(M(A))$ as a weak* dense subspace. Thus (d) is proved. Since multiplication in T corresponds to the convolution of Dirac measures, (b) is an immediate consequence of (d). Finally, (c) follows from the equation $\varphi(\psi \circ \varphi(t)) = \varphi(t\psi(u))$, i.e., $\varphi(t) = \varphi(t)u$, since φ is injective on $\psi(S)$ and $t\psi(u) \in \psi(S)$.

EXAMPLES. The above theorem is applicable e.g. in two classical situations, where the algebra A is defined in terms of a locally compact abelian topological group G. If A is $L^1(G)$, the convolution algebra of Haar integrable functions on G, then as is well known [5, p. 3] the multiplier algebra $M(A)$ may be identified with the convolution algebra $M(G)$ of bounded regular Borel measures on G. In this case, S is the Bohr compactification of G and $\psi(S)$ is the kernel (i.e., minimal ideal) of T [11, p. 164].

Similarly, the theorem yields a connection between the structure semigroups of the convolution measure algebras

$$L^1(G_+) = \left\{ f \in L^1(G) \,\middle|\, \int_{G \setminus G_+} |f(x)|\, dx = 0 \right\}$$

and

$$M(G_+) = \{\mu \in M(G) \mid |\mu|(G \setminus G_+) = 0\} ,$$

where G_+ is a closed subsemigroup of G containing the neutral element of G and such that the interior of G_+ generates G and is dense in G_+. For $A = L^1(G_+)$ satisfies the hypotheses of the theorem (for example, S has an identity, since G_+ may be realixed as a dense subsemigroup of S [11, p. 163]), and Birtel has shown in [1, p. 821] that $M(A) = M(G_+)$. In the case of $A = L^1(G_+)$ (and hence if $A = L^1(G)$) the usual order in $M(A)$ as a space of measures agrees with the order defined before Theorem 3.1, for it follows from Birtel's proof that there is a net $\{\mu_\sigma\}$ of positive $\mu_\sigma \in L^1(G_+)$ satisfying $\lim_\sigma \mu_\sigma * \mu(f) = \mu(f)$ for all $f \in C_0(G)$, $\mu \in M(G_+)$.

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