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ON THE EXISTENCE OF SUPPORT POINTS OF SOLID CONVEX SETS

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Let E be a separable Fréchet lattice. It is shown that a solid convex set X with void interior in E is supported at each of its boundary points if and only if the span of X is not dense in E. This result then is applied to the case of solid convex sets with void interior in real Fréchet spaces with an unconditional Schauder basis and in the real Banach lattice $C(S)$, S compact Hausdorff.

1. Introduction. If E is a real Hausdorff topological vector space and X is a convex subset of E with nonempty interior and boundary ∂X then, by a known theorem, every point of ∂X supports X, that is, for every $x \in \partial X$ there exists a continuous nontrivial linear functional f on E such that $\sup f(X) = f(x)$. However, if X has void interior, there are examples of compact convex sets, e.g., the Hilbert cube in l_2 [1, p. 160], which have boundary points that are not support points.

The object of this note is to investigate conditions on convex sets X with void interior in a separable real Fréchet lattice E, such that every point of ∂X is a support point of X. A theorem obtained is that for such sets X which are also solid, X is supported at each boundary point if and only if the span sp X of X is not dense in E. Moreover, if E is a real Fréchet space with an unconditional basis $\{x_n, f_n\}$ (the sequence space s, the Banach spaces c_0 and $l_p (1 \leqq p < \infty)$ and so all separable real Hilbert spaces are examples of such spaces) and if E is equipped with the ordering induced by the basis $\{x_n, f_n\}$, then a solid convex set X with void interior in E is supported at each of its boundary points if and only if $\bar{X}$ does not contain a weak order unit of E. On the other hand, if E is the Banach lattice $C(S)$, S compact Hausdorff, all solid convex subsets X with void interior in E have the property that the boundary points and the support points of X coincide.

2. Support properties of solid convex sets with void interior. A set X in a Fréchet lattice E is said to be *solid* if y is in X whenever x is in X and $|y| \leqq |x|$. An element x in the positive cone of E is said to be a *weak order unit* of E if $y = 0$ whenever y is in E and $x \wedge |y| = 0$. For the terminology see also H. H. Schaefer [5] or A. L. Peressini [4]. The topological boundary of X is denoted by ∂X.

THEOREM 1. *Let X be a solid convex set with void interior in a separable real Fréchet lattice E. Then every $x \in \partial X$ supports X if and only if* sp X *is not dense in* E.

Proof. (Sufficiency) Let E' be the topological dual of E. If sp X is not dense in E there is an $f \in E' \backslash \{0\}$ such that $f(X) = \{0\}$. In this case f obviously is a supporting functional of X for every $x \in \partial X$.

(Necessity) Let K be the positive cone of E and $K' = \{f \in E' : f(x) \geqq 0, x \in K\}$ the dual cone in E'. We define the sets S_x, $x \in X \cap K$, by

$$S_x = \{f \in K' : f(x) = 0\} .$$

It is clear that each S_x is a $\sigma(E', E)$-closed set in E' which contains 0. Moreover, let

$$S = \bigcap_{x \in X \cap K} S_x .$$

Since E is a Fréchet space there exists a countable base $\{U_n\}$ of neighborhoods of 0 in E, and since E is separable there is a sequence $\{V_n\}$ of open $\sigma(E', E)$-neighborhoods of 0 in E' satisfying $\bigcap_{n=1}^{\infty} V_n = \{0\}$. (The sequence $\{V_n\}$ can, for instance, be constructed in the following way: If $\{x_n\}$ is a dense set in E, let W_{mn} be defined by $W_{mn} = \{f \in E' : |f(x_n)| < 1/m\}$. It then follows that $\bigcap_{m,n=1}^{\infty} W_{mn} = \{0\}$ since for each f in this last intersection one has $f(\{x_n\}) = \{0\}$ and hence $f = 0$.) We assume now that $S = \{0\}$. Then $E' \backslash \{0\} = \complement S$ and so for all $m, n \in N$ one has

$$U_n^0 \subset \bigcup_{k=1}^{\infty} U_k^0 = E' = V_m \cup \bigcup_{x \in X \cap K} \complement S_x .$$

Since the polars U_n^0 of U_n are $\sigma(E', E)$-compact there is for each m and each n in N a finite set A_{mn} in $X \cap K$ such that

$$U_n^0 \subset V_m \cup \bigcup_{x \in A_{mn}} \complement S_x .$$

If $\{x_k\}$ is a sequence in $X \cap K$ such that $\{x_k\} = \bigcup_{m,n=1}^{\infty} A_{mn}$ we get for all $m \in N$,

$$E' = \bigcup_{n=1}^{\infty} U_n^0 = V_m \cup \bigcup_{k=1}^{\infty} \complement S_{x_k} .$$

Whence

$$E' = \bigcap_{m=1}^{\infty} \left(V_m \cup \bigcup_{k=1}^{\infty} \complement S_{x_k} \right) = \bigcup_{k=1}^{\infty} \complement S_{x_k} \cup \{0\}$$

and

$$(1) \qquad \bigcap_{k=1}^{\infty} S_{x_k} = \complement \left(\bigcup_{k=1}^{\infty} \complement S_{x_k} \right) = \complement (E' \backslash \{0\}) = \{0\} .$$

Next, if d is a translation invariant metric generating the topology of E, we define the real sequence $\{a_k\}$ by

$$a_k = \inf \{2^{-k}, \sup \{t > 0: d(0, sx_k) \leqq 2^{-k}, s \in [0, t]\}\} .$$

Since X is solid we have $(2^n - 1)2^{-n+k} a_k x_k \in X$ for all $k, n \in N$ and since X is convex,

$$\sum_{k=1}^{n} a_k x_k = (2^n - 1)^{-1} 2^n \sum_{k=1}^{n} 2^{-k}(2^n - 1)2^{-n+k} a_k x_k \in X$$

for all $n \in N$. Since $\bar{X}$ is complete and since for $n < m$

$$d\left(\sum_{k=1}^{n} a_k x_k, \sum_{k=1}^{m} a_k x_k\right) \leqq \sum_{k=n+1}^{m} d(0, a_k x_k) \leqq \sum_{k=n+1}^{m} 2^{-k} < n^{-1} ,$$

$\lim_n \sum_{k=1}^{n} a_k x_k$ exists in $\bar{X}$ and this limit is denoted by x. Since int $X = \varnothing$ and $\bar{X}$ is solid [4, Proposition 2. 4. 8] it follows that $1/2x \in \partial X$ and thus is a support point of X. If f is a corresponding support functional we have $f \neq 0$ and $0 = f(0) \leqq f(1/2x) = f(x) - f(1/2x)$. If $\{y_n\} \subset X$ is a sequence that converges to x in E one obtains $f(x) \leqq \sup_n f(y_n) \leqq f(1/2x)$, and hence $f(x) = 0$. Now, since E is a locally convex lattice and again since $\bar{X}$ is solid, it follows that

$$\begin{aligned} 0 &\leqq |f|(x) \\ &\leqq \sup \{f(y): y \in E, |y| \leqq x\} \leqq \sup f(\bar{X}) = \sup f(X) = f\left(\frac{1}{2}x\right) = 0 . \end{aligned}$$

Therefore, $|f| \in K' \backslash \{0\}$ and $|f|(x) = 0$. This shows that

$$S_x \backslash \{0\} \neq \varnothing .$$

Since $0 \leqq a_k x_k \leqq x$, one has $0 \leqq g(x_k) \leqq a_k^{-1} g(x) = 0$, $g \in S_x$, $k \in N$. In view of (1) one thus obtains

$$\{0\} \subset S_x \subset \bigcap_{k=1}^{\infty} S_{x_k} = \{0\} ,$$

and this contradiction shows that

$$S \backslash \{0\} \neq \varnothing .$$

If f is a nonzero element of S then $f(X \cap K) = \{0\}$. Thus for any $x \in X$ we have $f(x) = f(x^+) - f(x^-) = 0$ because $x^\pm \in X$. Hence $f(X) = \{0\}$, showing that sp X cannot be dense in E.

Let now E be a real Fréchet space with an unconditional basis $\{x_n, f_n\}$. It is known that the set $K = \{x \in E: f_n(x) \geqq 0, n \in N\}$ is a closed, normal, generating cone in E and equipped with K, E becomes an order complete locally convex lattice [3, Theorem 5]. Obviously, $\{x_n\} \subset K$ and the coefficient functionals f_n are positive with respect to K. Therefore, the basis is a positive Schauder basis for E [2].

REMARK. A slight modification of the above argument shows that in Theorem 1 the separability of E can be replaced by the (weaker) condition: There exists a sequence $\{u_n\}$ in the positive cone of E such that $\operatorname{sp} \bigcup_{n=1}^{\infty} [0, u_n]$ is dense in E.

THEOREM 2. *If X is a solid convex set with void interior in E (E being specified above), then every point of ∂X supports X if and only if $\bar{X}$ does not contain a weak order unit of E.*

Proof. (Necessity) Let every point of ∂X support X and let us assume that $\bar{X}$ contains a weak order unit x of E. Then from [3, Proposition 11] it follows that the span of $[0, x]$ is dense in E. Since $[0, x] \subset \bar{X}$ this contradicts Theorem 1. Hence $\bar{X}$ does not contain a weak order unit.

(Sufficiency) If $\bar{X}$ does not contain a weak order unit of E, suppose that $\sup f_n(X) > 0$, $n \in N$. Then for every n there is a $y_n \in X$ such that $\sup f_n(X) \leqq 2 f_n(y_n)$. Since X is solid this yields for all n that $\sup f_n(X) x_n \leqq 2 \mid y_n \mid$; whence $1/2 \sup f_n(X) x_n \in X$. In the same way as in the proof of the necessity part of the preceding theorem we can now construct an element $x \in \bar{X} \cap K$ such that $x = \lim_n \sum_{i=1}^{n} a_i x_i$, where $a_i > 0$, $i \in N$. If $y \in K \backslash \{0\}$ then there must be a positive integer n such that $f_n(y) > 0$. If $z \in K$ is given by $z = \inf \{a_n, f_n(y)\} x_n$ it follows that $z \neq 0$ and $z = x \wedge y$, i.e., x is a weak order unit of E in $\bar{X}$. By this contradiction to our assumption there is an $n \in N$ such that $\sup f_n(X) = 0$. Therefore, $\operatorname{sp} X$ cannot be dense in E and an application of Theorem 1 finally completes the proof.

Concerning the real Banach lattice $C(S)$, S compact Hausdorff, it is clear that there can exist solid subsets X of $C(S)$ with void interior containing a weak order unit of $C(S)$ and such that every boundary point of X is a support point of X. For instance, take $X = \{y \in C[0, 1] : \mid y \mid \leqq x\}$, where x, given by $x(s) = s$, $s \in [0, 1]$, is a weak order unit of $C[0, 1]$. Therefore, that $\bar{X}$ contains no weak order unit of $C(S)$ is not a necessary condition for X to be supported at each boundary point, as is also seen by the following theorem:

THEOREM 3. *If X is a convex solid set with void interior in $C(S)$ then every boundary point of X supports X.*

Proof. We assume that $\operatorname{sp} X$ is dense in $C(S)$. If f_s is the point evaluation functional of a general point s of S, this implies that $\sup f_s(X) > 0$, $s \in S$. In this case, since X is solid, there is for every $s \in S$ an $x_s \geqq 0$ in X such that $x_s(s) > 0$. Hence for every $s \in S$

there is an open neighborhood V_s of s in S such that $\inf x_s(V_s) > 0$. Since S is compact and $\{V_s\colon s \in S\}$ is an open covering of S there is a finite subcovering $\{V_{s(1)}, \cdots, V_{s(m)}\}$ for S. Taking $x = m^{-1} \sum_{n=1}^{m} x_{s(n)}$ it is clear that x is in X since X is convex, and that

$$\inf x(S) \geqq m^{-1} \inf_{n \leqq m} \inf x_{s(n)}(V_{s(n)}) > 0 .$$

If U is the unit ball of $C(S)$ we obtain $(\inf x(S)) \, |y| \leqq x$, $y \in U$, which, since X is solid, implies that $(\inf x(S)) U \subset X$. This contradiction shows that $\overline{\mathrm{sp}}\, X \neq C(S)$ and the result follows in the same way as in the sufficiency part of the proof of Theorem 1.

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