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STOPPING TIMES FOR BERNOULLI AUTOMORPHISMS

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The purpose of this note is to study a certain class of stopping times for Bernoulli automorphisms by means of the Friedman-Ornstein results concerning weakly Bernoulli partitions.

1. **Introduction.** Let T be an automorphism of the non-atomic Lebesgue space $(X, \mathfrak{A}, \mu)$ and let $\theta: X \rightarrow \boldsymbol{Z}^+$ be a measurable function. If the transformation $S = T^\theta$ defined by $S(x) = T^{\theta(x)}(x)$, for $x \in X$, is an automorphism of X then θ is called a stopping time for T. Such a stopping time will be said to be of nth order (where n is a positive integer or ∞) if there exists a decreasing sequence $D_1 \supset D_2 \supset D_3 \supset \cdots$ of measurable subsets of X satisfying

(a) $\mu(D_n) > 0$ and $\mu(D_{n+1}) = 0$ if n is finite

or

(b) $\mu(D_i) > 0$ for all i and $\mu(\bigcap_1^\infty D_i) = 0$ if n is infinite

such that T^θ coincides (modulo 0) with the automorphism M defined by

$$M(x) = T_{D_0} \circ T_{D_1} \circ T_{D_2} \circ \cdots \circ T_{D_{s-1}} \circ T_{D_s}(x) \,, \qquad \text{for } x \in D_s - D_{s+1} \,,$$

for $s = 0, 1, 2, \cdots, n-1$, where $D_0 = X$ and T_{D_i} denotes the automorphism induced by T on D_i.

Neveu has shown [3] that every stopping time θ for which T^θ is ergodic is an nth order stopping time for a unique n. Moreover, the sets $D_1, D_2, \cdots$ are also unique (modulo 0). It follows from the work of Belinskaya [1] that if θ is an nth order stopping time for T then $h(T^\theta) = n\, h(T)$.

The purpose of this note is to study certain ergodic properties of T^θ under the hypothesis that T be a Bernoulli automorphism. For definitions and notation of entropy theory the reader is referred to [4] and [6]. For convenience of notation we shall let $P_n^m = \bigvee_n^m T^iP$, $P^+ = \bigvee_0^\infty T^iP$, and $P^- = \bigvee_{-\infty}^0 T^iP$ where $m > n$ and P is a partition of X.

2. We now establish a result, using a technique developed in [7], concerning a special class of stopping times for a Bernoulli automorphism.

Let T be a Bernoulli automorphism of X, and let B be a Bernoulli partition for T, i.e., B is a generator and the orbit of B under T, $\{T^iB: i \in \boldsymbol{Z}\}$, is a jointly independent sequence of partitions. We let $\mathscr{F}_B$ denote the collection of all measurable partitions P of X for which $H(P^+ \mid B^+) + H(P^- \mid B^-) < \infty$.

THEOREM. *Let T be a Bernoulli automorphism of X and B be a Bernoulli partition. Let θ be an nth order stopping time for T and let $D_1, D_2, \cdots$ be the sets corresponding to θ. Let P denote the partition $\{X - D_1, D_i - D_{i+1}: i = 1, 2, \cdots\}$. Suppose $P \in \mathscr{F}_B$. Then $S = T^\theta$ is weakly mixing if and only if S is a Bernoulli automorphism having entropy $n\,h(T)$.*

To prove this theorem we will require the following lemma.

LEMMA 1. *Let A, F and C be measurable partitions of X such that F is independent of A and $H(C \mid A) < \varepsilon$. Let $D \leqq C$ be finite. Then*

$$H(D \mid F) \geqq H(D) - \varepsilon .$$

Proof. Choose $F_n \leqq F$ such that $F_n \uparrow F$ and $H(F_n) < \infty$. Then

$$H(F_n \mid D) \geqq H(F_n \mid A) - H(D \mid A) \geqq H(F_n) - \varepsilon .$$

Hence

$$H(D \mid F_n) \geqq H(D) - \varepsilon .$$

Letting $n \to \infty$ we obtain the desired result.

Proof of theorem. Let K be any positive integer, $\varepsilon > 0$, and $Q = B_{-K}^{K}$ (this notation will be employed only with respect to the automorphism T). Choose $N > 0$ such that $H(P_{-\infty}^{0} \mid B_{-\infty}^{N}) < \varepsilon/4$ and $H(P_0^{\infty} \mid B_{-N}^{\infty}) < \varepsilon/4$. Let $R = \max\{N, K\}$. If S is weakly mixing then there is an integer $M > R$ for which

$$H(S^{-M}Q \mid B_{-R}^{R}) \geqq H(S^{-M}Q) - \frac{\varepsilon}{4} .$$

Since

$$S^{-M}Q \leqq B_{-\infty}^{K} \vee P_{-\infty}^{0}$$

and

$$H(B_{-\infty}^{R} \vee P_{-\infty}^{0} \mid B_{-\infty}^{R}) < \frac{\varepsilon}{4} ,$$

Lemma 1 implies that

$$H(S^{-M}Q \vee B_{-R}^{R} \mid B_{R+1}^{\infty}) \geqq H(S^{-M}Q \vee B_{-R}^{R}) - \frac{\varepsilon}{4} .$$

Using the fact that

$$\bigvee_0^\infty S^j Q \leqq B_{-K}^{\infty} \vee P_0^\infty$$

we obtain:

$$\begin{aligned}
H\Big(S^{-M}Q \Big| \bigvee_0^\infty S^j Q\Big) &\geqq H(S^{-M}Q \mid B_{-R}^{\infty} \vee P_0^\infty) \\
&\geqq H(S^{-M}Q \mid B_{-R}^{\infty}) - H(P_0^\infty \mid B_{-R}^{\infty}) \\
&\geqq H(S^{-M}Q \mid B_{-R}^{\infty}) \ - \frac{\varepsilon}{4} \\
&= H(S^{-M}Q \vee B_{-R}^{R} \mid B_{R+1}^{\infty}) - H(B_{-R}^{R} \mid B_{R+1}^{\infty}) - \frac{\varepsilon}{4} \\
&\geqq H(S^{-M}Q \vee B_{-R}^{R}) - H(B_{-R}^{R}) - \frac{\varepsilon}{2} \\
&= H(S^{-M}Q \mid B_{-R}^{R}) - \frac{\varepsilon}{2} \\
&\geqq H(S^{-M}Q) - \varepsilon\,.
\end{aligned}$$

Since K and ε were arbitrary, there exists an integer $p > N$ for which

$$H\Big(S^{-p}(B_{-R}^{R}) \Big| \bigvee_0^\infty S^j(B_{-R}^{R})\Big) \geqq H(B_{-R}^{R}) - \frac{\varepsilon}{2}\,.$$

Now, for all $t > p$,

$$\begin{aligned}
H\Big(\bigvee_p^t S^i Q \Big| \bigvee_{-\infty}^0 S^j Q\Big) &\geqq H\Big(\bigvee_p^t S^i Q \mid B_{-\infty}^{R} \vee P_{-\infty}^{0}\Big) \\
&\geqq H\Big(\bigvee_p^t S^i Q \mid B_{-\infty}^{R}\Big) - H(P_{-\infty}^{0} \mid B_{-\infty}^{R}) \\
&\geqq H\Big(\bigvee_p^t S^i Q \mid B_{-\infty}^{R}\Big) - \frac{\varepsilon}{4} \\
&= H\Big(\bigvee_p^t S^i Q \mid B_{-R}^{R} \vee B_{-\infty}^{-R-1}\Big) - \frac{\varepsilon}{4} \\
&= H\Big(\bigvee_p^t S^i Q \vee B_{-R}^{R} \mid B_{-\infty}^{-R-1}\Big) - H(B_{-R}^{R}) - \frac{\varepsilon}{4} \\
&\geqq H\Big(\bigvee_p^t S^i Q \vee B_{-R}^{R}\Big) - \frac{\varepsilon}{2} - H(B_{-R}^{R}) \\
&\geqq H\Big(\bigvee_p^t S^i Q\Big) + H(B_{-R}^{R}) - \varepsilon - H(B_{-R}^{R}) \\
&= H\Big(\bigvee_p^t S^i Q\Big) - \varepsilon\,.
\end{aligned}$$

This verifies that Q is a weakly Bernoulli partition for S and thus, applying the Friedman-Ornstein theorem [2], Q generates a Bernoulli factor. As a result of Ornstein's theorem 2 of [5], letting $K \to \infty$, we find that S is actually a Bernoulli automorphism.

3. We now illustrate some consequences of the theorem in the case of second order stopping times.

We omit the proofs of the following two elementary lemmas.

LEMMA 2. *If R is an automorphism of X and A is a measurable subset of X for which $\bigcup_0^\infty R^i A = X$ then R is ergodic if and only if R_A is ergodic.*

LEMMA 3. *Let R be an automorphism of X. If R^2 is weakly mixing then so is R.*

PROPOSITION 1. *Let T be an automorphism of X and let θ be a second order stopping time for T. Then $S = T^\theta$ is ergodic if and only if both T and $(T_{D_1})^2$ are ergodic.*

Proof. If $S = T^\theta$ is ergodic it is well-known that $S_{D_1} = (T_{D_1})^2$ is also ergodic. Hence T_{D_1} is ergodic. From $\bigcup_0^\infty S^i D_1 = X$ it follows that $\bigcup_0^\infty T^i D_1 = X$. Applying Lemma 2 we obtain the ergodicity of T.

Conversely suppose T and $(T_{D_1})^2$ are ergodic. In view of Lemma 2, it suffices to show $\bigcup_0^\infty S^i D_1 = X$. One easily verifies that

$$S\Big(\bigcup_0^n T^i D_1\Big) \cup D_1 = \bigcup_0^{n+1} T^i D_1 \qquad (\text{for } n \geqq 0)$$

from which is obtained $\bigcup_0^\infty S^i D_1 = \bigcup_0^\infty T^i D_1 = X$.

COROLLARY 1. *Let T be a Bernoulli automorphism of X, B be a Bernoulli partition for T, and θ be a second order stopping time defined by choosing D_1 to be an atom of $\bigvee_{-K}^{K} T^i B$ for any integer K. Then $S = T^\theta$ is ergodic.*

Proof. It follows from a corollary of Theorem 1 of [7] that T_{D_1} is Bernoulli and hence, of course, $(T_{D_1})^2$ is ergodic. Thus Proposition 1 yields that S is ergodic.

PROPOSITION 2. *Let T be a Bernoulli automorphism of X and B be a Bernoulli partition for T. Let θ be a second order stopping time for which $\{D_1, X - D_1\} \in \mathscr{F}_B$. Then the following are equivalent:*

(a) *T_{D_1} is weakly mixing.*

(b) *T_{D_1} is Bernoulli.*

(c) *S_{D_1} is Bernoulli.*

(d) *S_{D_1} is weakly mixing.*

Proof. Using Lemma 3 together with the observation that $S_{D_1} = (T_{D_1})^2$ and Theorem 1 of [7] the proof is immediate.

PROPOSITION 3. *Let T be a Bernoulli automorphism of X and $B = \{B^1, B^2, \cdots, B^K\}$ be a Bernoulli partition for T. Let θ be the second order stopping time for T defined by choosing $D_1 = B^1$. Then $S = T^\theta$ is mixing.*

Proof. Let K be a fixed positive integer and set $Q = B^K_{-K}$. As a consequence of the definition of S one can verify that

$$\bigvee_K^\infty S^i Q \leqq B_1^\infty \quad \text{and} \quad \bigvee_{-\infty}^{-K} S^i Q \leqq B^0_{-\infty} .$$

So if A and B are members of the algebra generated by the atoms of Q then $\mu(S^j A \cap B) \rightarrow \mu(A)\mu(B)$. Now a standard approximation argument will show that S is mixing.

COROLLARY 2. *Under the hypotheses of Proposition 3 the automorphism $S = T^\theta$ is Bernoulli.*

Proof. This follows immediately from our theorem.

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