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SOME REMARKS ON HARMONIC MEASURE IN SPACE

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The purpose of this paper is to examine the relationship between harmonic measure and $n-1$ dimensional Hausdorff measure for a class of domains in R^n with irregular boundaries. It is shown for these domains that harmonic measure and Hausdorff measure have the same null sets.

This investigation was motivated in part by the work of Hunt and Wheeden, [5], [6]. In these papers they consider Lipschitz domains, that is, domains whose boundaries are locally representable by graphs of Lipschitz functions. One of their main results is that a positive harmonic function defined on a Lipschitz domain has a nontangential limit at all points of the boundary except possibly those that belong to a set of harmonic measure zero. In the classical case where the domain is taken to be the half-space of R^n, the nontangential limit is known to exist at H^{n-1} almost every point of the boundary, c.f., [2], [3]. We will show that for domains Ω satisfying a geometric measure theoretic condition, H^{n-1} (restricted to the boundary of Ω) and harmonic measure have the same null sets. Therefore, for these domains, the results of Hunt and Wheeden will represent a generalization of the classical case.

By use of the conformal mapping theorem it is not difficult to prove, for a domain in R^2 whose boundary is a simple closed rectifiable curve, that harmonic measure and H^1 measure have the same null sets. In § 4 it will be shown that the analog of this does not hold in R^3. We give an example of a topological 2-sphere whose boundary has finite H^2 measure and has a tangent plane at each point, but for which H^2 measure is not absolutely continuous with respect to harmonic measure.

2. Preliminaries. Let Ω be a bounded open subset of R^n and consider the Banach space $C(\partial\Omega)$, the space of continuous functions on the compact set $\partial\Omega$ with the norm given by $\sup\{|f(y)|: y \in \partial\Omega\}$, $f \in C(\partial\Omega)$. For each $x \in \Omega$, let $\lambda_x: C(\partial\Omega) \to R^1$ be the bounded linear functional defined by $\lambda_x(f) = u_f(x)$, where u_f is the harmonic function corresponding to the boundary values f. Hence, there is a unit measure μ_x on $\partial\Omega$ called *harmonic measure*, such that

$$\lambda_x(f) = u_f(x) = \int f d\mu_x ,$$

$f \in C(\partial\Omega)$. If G is a component of Ω, the class of Borel subsets of

$\partial\Omega$ of μ_x measure zero is independent of $x \in \Omega$. Moreover, the class of μ_x integrable functions is also independent of x.

The domains under consideration in this paper are contained in the class of sets of *finite perimeter*. A measurable set $E \subset R^n$ is said to have finite perimeter if the gradient of its characteristic function, when taken in the sense of distributions, is a totally finite vector-valued measure. A unit vector n is called the exterior normal (in measure theoretic sense) to E at y if the symmetric difference of E and the half-space

$$R^n \cap \{x: (x - y) \cdot n < 0\}$$

has density 0 at y. The set of points y where $n = n(y)$ exists is called the *reduced boundary* of E and is denoted by $\partial^* E$. If E has finite perimeter, then $H^{n-1}(\partial^* E) < \infty$ and

$$\int_E \operatorname{div} \zeta dL_n(x) = \int_{\partial^* E} \zeta(y) \cdot n(y) dH^{n-1}(y) \tag{1}$$

for every smooth vector field ζ. Here, L_n denotes n-dimensional Lebesgue measure. Proofs of these facts concerning sets of finite perimeter can be found in [4].

DEFINITION 2.1. For a measurable set $E \subset R^n$, $B \subset \partial E$, and $x \in R^n$, let the *variation of B at x* be defined by

$$v(B, x) = \int_{\partial^* E \cap B} \frac{|n(y) \cdot (y - x)|}{|y - x|^n} dH^{n-1}(y) .$$

If $v(B, x) < \infty$, then the following is meaningful:

$$s(B, x) = \int_{\partial^* E \cap B} \frac{n(y) \cdot (y - x)}{|x - y|^n} dH^{n-1}(y) .$$

In the event E is a domain with smooth boundary, then $v(\partial E, x)$ can be regarded as the area of the radial projection

$$p_x : \partial E \longrightarrow S^{n-1} ,$$

and $s(\partial E, x)$ reduces to the notion of the solid angle.

By means of the Gauss-Green Theorem as given in (1), it can be easily verified that in case E is a bounded open set with finite perimeter, then

$$\begin{aligned} s(\partial E, x) &= \omega_{n-1} , & x &\in E \\ s(\partial E, x) &= 0 , & x &\in R^n - \bar{E} . \end{aligned}$$

Where ω_{n-1} is the H^{n-1} measure of the unit sphere in R^n.

Sets for which $v(\partial E, x)$ is a bounded function were investigated by Kral, [7], for the purpose of giving a geometric meaning to the normal derivative (taken in the sense of distributions) of the Newtonian potential.

3. Domains with boundaries having finite variation. Let $\Omega \subset R^n$ be an open, bounded, connected set, with $\partial(\operatorname{int}\Omega) = \partial(R^n\text{-Clos}\,\Omega)$ and let $p \in \Omega$. Then the Green's function for Ω can be written as

$$G(p, x) = I_2 * \delta(p) - I_2 * \mu_p \tag{2}$$

where $I_2(x) = ((n-2)\omega_{n-1})^{-1} |x|^{2-n}$ and $\delta(p)$ denotes the unit measure concentrated at p. For p fixed in Ω, $G_p(x) = G(p, x)$ can be defined for all $x \notin \Omega$ and consequently if $\varphi \in C^\infty(R^n)$ with $p \notin \operatorname{spt}\varphi$,

$$\Delta G_p(\varphi) = \mu_p(\varphi) ,$$

where Δ is the Laplacian taken in the sense of distributions. On the other hand, if Ω_k is a sequence of domains with smooth boundaries invading Ω, we have

$$\begin{aligned} \Delta G_p(\varphi) &= \int_\Omega G_p \Delta\varphi dL_n = \lim_{k\to\infty}\int_{\Omega_k} G_p\Delta\varphi dL_n \\ &= -\lim_{k\to\infty}\int_{\partial\Omega_k} \varphi\nabla G_p \cdot n dH^{n-1} \\ &= -\lim_{k\to\infty}\int_{\Omega_k} \nabla\varphi\cdot\nabla G_p dL_n \\ &= -\int_\Omega \nabla\varphi\cdot\nabla G_p dL_n . \end{aligned}$$

Therefore,

$$\int_\Omega \nabla G_p \cdot \nabla\varphi = -\mu_p(\varphi) \tag{3}$$

whenever $p \notin \operatorname{spt}\varphi$. If h is a harmonic function in Ω whose gradient is integrable over Ω, then the generalized normal derivative (using the exterior normal) of h is defined by

$$Nh(\varphi) = \int_\Omega \nabla h \cdot \nabla\varphi dL_n$$

for all $\varphi \in C_0^\infty(R^n)$. We shall take the generalized normal derivative of the right side of (2).

First, it is elementary that

$$N(I_2 * \delta(p))(\varphi) = -1/\omega_{n-1}\int_{\partial^*\Omega} n(y)\cdot\frac{(y-p)}{|y-p|^n}\varphi(y)dH^{n-1}(y) ,$$

and as $p \in \Omega$, $N(I_2 * \delta(p))$ is a measure.

Now in order to consider $N(I_2{}^*\mu_p)$ we impose the additional hypothesis

$$\int v(\partial\Omega,\, y)d\mu_p(y) < \infty\ . \tag{4}$$

Recall that this condition is independent of p. For any $y \in \partial\Omega$, observe that

$$N(I_2{}^*\delta(y))(\varphi) = 1/\omega_{n-1}\int_\Omega \nabla\varphi(x)\cdot\frac{(y-x)}{|\,y-x\,|^n}dL_n(x)\ . \tag{5}$$

Also, an application of Fubini's Theorem provides

$$N(I_2{}^*\mu_p)(\varphi) = \int N(I_2{}^*\delta(y))(\varphi)d\mu_p(y)\ . \tag{6}$$

In [7], Kral investigated conditions under which $N(I_2{}^*\mu)$ is a measure whenever μ is a measure supported by $\partial\Omega$. For this purpose he proved

$$\begin{aligned}&\sup\left\{\int_\Omega \nabla\varphi(x)\cdot\frac{y-x}{|\,y-x\,|^n}dL_n(x)\colon \varphi\in C_0^\infty(R^n),\ |\,\varphi\,|\leqq 1\right\}\\ &\leqq \omega_{n-1} + v(\partial\Omega,\, y)\ .\end{aligned}$$

Therefore, in particular, it follows from (4) and (5) that $N(I_2{}^*\mu_p)$ is a measure. In view of the fact that Ω is assumed to be of finite perimeter, it follows from the definition of the measure theoretic exterior normal that the Lebesgue n-dimensional density of Ω is equal to 1/2 at H^{n-1} almost all points of $\partial^*\Omega$. This allows us to conclude from [7, Lemma 3.2] that

$$N(I_2{}^*\delta(y))(f) = \frac{f(y)}{2} - \frac{1}{\omega_{n-1}}\int_{\partial^*\Omega} f(x)n(x)\cdot\frac{(x-y)}{|\,x-y\,|^n}dH^{n-1}(x)$$

whenever f is a bounded Baire function on $\partial\Omega$. Thus, if $E \subset \partial\Omega$ is a Borel set, it now follows from (6) that

$$N(I_2{}^*\mu_p)(E) = \mu_p(E)/2 - 1/\omega_{n-1}\int s(E,\, y)d\mu_p(y)\ . \tag{7}$$

From (3), (2), and (7) we obtain

$$-\mu_p(E) = -1/\omega_{n-1}s(E,\, p) - \left[\mu_p(E)/2 - 1/\omega_{n-1}\int s(E,\, y)d\mu_p(y)\right]$$

or

$$\omega_{n-1}/2\mu_p(E) = s(E,\, p) - \int s(E,\, y)d\mu_p(y)\ .$$

Thus, in summary, we have the following.

THEOREM 3.2. *Let Ω be a bounded, open, connected set of finite perimeter with $\partial(\operatorname{int}\Omega) = \partial(R^n - \operatorname{Clos}\Omega)$. If $p \in \Omega$ and*

$$\int v(\partial\Omega, y)d\mu_p(y) < \infty ,$$

then

$$\frac{\omega_{n-1}}{2}\mu_p(E) = s(E, p) - \int s(E, y)d\mu_p(y)$$

for every Borel set $E \subset \partial\Omega$.

COROLLARY 3.3. *If $E \subset \partial\Omega$ is a Borel set with $H^{n-1}(\partial^*\Omega \cap E) = 0$, then $\mu_p(E) = 0$.*

The proof is obvious since $s(E, x) = 0$ whenever $H^{n-1}(\partial^*\Omega \cap E) = 0$ and $x \in R^n$.

We will now find conditions under which H^{n-1} restricted to $\partial^*\Omega$ is absolutely continuous with respect to harmonic measure μ_p. To this end we first establish

LEMMA 3.4. *Suppose Ω is a set of finite perimeter and let $E \subset \partial^*\Omega$ be a Borel set of positive H^{n-1} measure. There is a set $F \subset E$ with $H^{n-1}(E - F) = 0$ satisfying the following conditions: if $\varepsilon > 0$, $y \in F$, and if $p \in \Omega$ is on the interior normal to Ω at y, then there is a set $F_p \subset F$ containing y such that*

$$\lim_{p \to y} (\operatorname{diam} F_p) = 0$$

and

$$\lim_{p \to y} s(F_p, p) > \omega_{n-1}/2 - \varepsilon .$$

Proof. We refer the reader to Chapters 3 and 4 of [4] for proofs of the following facts. The set $\partial^*\Omega$ is $(H^{n-1}, n-1)$ rectifiable and therefore, there is a countable number of class C^1 manifolds of dimension $n - 1$, M_i, such that

$$H^{n-1}(\partial^*\Omega - \bigcup_{i \geqq 1} M_i) = 0 .$$

Moreover, for H^{n-1} a.e. $y \in \partial^*\Omega$, the exterior normal to Ω at y is orthogonal to the tangent plane of some M_i at y. Finally, for H^{n-1} a.e. $y \in E$,

$$\lim_{r\to 0} H^{n-1}(B(y, r) \cap E \cap M_i)/\alpha(n-1)r^{n-1} = 1 \tag{7}$$

for some M_i. Here $\alpha(n-1)$ denotes the volume of the unit ball in R^{n-1} and $B(y, r)$ is the n-ball of radius r centered at y. Let F denote the complement in E of the union of the three H^{n-1} null sets just described.

Now select $y \in F$ and $\varepsilon > 0$. Then $y \in M_i$ for some i. For convenience, call $M_i = M$, take $y = 0 \in R^n$, and assume the tangent plane to M at 0 is R^{n-1}. Let n denote the exterior normal at $y = 0$ and for points p on the interior normal at $y = 0$, write $p = (0, v_p)$, $0 \in R^{n-1}$, $v_p < 0$. Let $\eta = \eta(\varepsilon)$ be a number to be determined below and consider the cone

$$C = \{x: n \cdot x > \eta \,|\, x \,|\} .$$

For $x \in R^n$, write $x = (u, v)$, $u \in R^{n-1}$, $v \in R^1$ and let

$$C_p = \{(u, v): v = w + v_p,\ (u, w) \in C\} ,$$

the cone with vertex at p. Since M is a manifold of class C^1, it is clear that for η sufficiently small,

$$\lim_{p\to 0} s(M_p, p) > \omega_{n-1}/2 - \varepsilon , \tag{8}$$

where $M_p = M \cap C_p$. Let $A = M - F$, $F_p = F \cap C_p$, and $A_p = A \cap C_p$. Then

$$|\, s(A_p, p) \,| \leqq \int_{A_p} |\, y - p \,|^{1-n} dH^{n-1}(y). \tag{9}$$

If $r_p = \sup\{|\, y \,|: y \in A_p\}$, then from (7)

$$\lim_{p\to 0} H^{n-1}(A_p) \cdot r_p^{1-n} = 0 . \tag{10}$$

Furthermore, since A is a subset of a class C^1 manifold, there is a constant K such that $|\, y - p \,| \geqq K r_p$ whenever $y \in A_p$ and $|\, p \,|$ is sufficiently small. The conclusion of the lemma now follows from (8), (9), and (10).

In the following theorem, a condition is given under which H^{n-1} restricted to $\partial^*\Omega$ is absolutely continuous with respect to harmonic measure. This condition is similar, although weaker, to the one introduced by Kral [7, § 3] to determine the Fredholm radius of an operator associated with the classical double layer potential.

THEOREM 3.5. *Let Ω be an open set satisfying the conditions of Theorem* 3.2. *In addition to the hypotheses of Theorem* 3.2, *assume that for H^{n-1}* a.e. *$y_0 \in \partial^*\Omega$, there exists a positive number $r = r(y_0)$ such that*

$$\limsup_{\substack{y \to y_0 \\ y \in \partial^* \Omega}} v[B(y, r) \cap \partial\Omega, y] < \omega_{n-1}/2 \ .$$

Then $H^{n-1}(E \cap \partial^*\Omega) = 0$ *whenever* $E \subset \partial\Omega$ *is a Borel set with* $\mu_p(E) = 0$.

Proof. Let $E \subset \partial\Omega$ be a Borel set with $\mu_p(E) = 0$ and assume that $H^{n-1}(E \cap \partial^*\Omega) > 0$. Let $F \subset E$ be the set associated with E as given by Lemma 3.4, and select $y_0 \in F$ that satisfies the hypotheses of our theorem. Then, with every $\varepsilon > 0$, there is a positive number $t < r(y_0)$ such that

$$v[B(y, t) \cap \partial\Omega, y] < \omega_{n-1}/2 - \varepsilon \ . \tag{11}$$

As $\mu_p(E) = \mu_p(F) = 0$, it follows from Theorem 3.2 that

$$s(F, p) = \int s(F, y) d\mu_p(y) \ . \tag{12}$$

Now let p approach y_0 along the interior normal and observe that Lemma 3.4 implies $F_p \subset B(y_0, t/4)$ for $|p - y_0|$ sufficiently small. Therefore, it follows from (11) that

$$|s(F_p, y)| < \omega_{n-1}/2 - \varepsilon$$

for all $y \in B(y_0, t/2)$. Moreover, $|s(F_p, y)| \to 0$ uniformly on $\partial\Omega - B(y_0, t/2)$ as $|p - y_0| \to 0$. In view of (12) and Lemma 3.4, this produces a contradiction. Therefore, $H^{n-1}(E \cap \partial^*\Omega) = 0$.

4. An example. In §3 we imposed certain regularity conditions on $\partial\Omega$ in order to conclude that harmonic measure and H^{n-1} restricted to $\partial^*\Omega$ have the same null sets. In R^2, harmonic measure and H^1 restricted to $\partial\Omega$ have the same null sets provided Ω is bounded by a simple closed curve of finite H^1 measure. In this section, we will show that the analog of this does not hold in R^3.

For the purpose of constructing this domain in R^3, we first consider a closed arc, C, in R^2 that occupies positive Lebesgue measure (or equivalently, H^2 measure) in R^2. It is not difficult to modify the standard Osgood construction of C in such a way so as to produce a set, S, of points on C of positive H^2 measure such that every point of S cannot be joined to any point interior to C by a rectifiable curve. Now let g be function of class C^1 defined on the interior of C with the properties

(i) $g > 0$, (ii) $|\nabla g|$ is bounded, (iii) $g < \delta$, where $\delta(x) =$ distance from x to C.
Define

$$f(x) = \exp(-1/g(x)) \ , \qquad x \in \text{interior } C \ ,$$

and let

$$\Omega = \{(x, y): x \in \text{interior } C,\ 0 \leqq y < f(x)\}\ .$$

Since C is a closed arc, it follows that $\partial\Omega$ is a topological 2-sphere. Moreover, $\partial\Omega$ has a tangent plane at each of its points and $H^2(\partial\Omega) < \infty$. On the other hand, $S \subset \Omega$ is a set of positive H^2 measure with the property that each of its points cannot be joined to any point of Ω by a rectifiable curve. Consequently, a result of Brelot and Choquet [1] states that the harmonic measure of S is zero.

In conclusion we would like to point out that there is no inclusion relationship between the class of smoothly bounded domains and the class of domains that satisfy the conditions of Theorem 3.5. However, if the normal of a smoothly bounded domain is locally Dini continuous, then Theorem 3.5 applies. We are indebted to Grant Welland for this observation.

To see this, let $f: U \to R^1$ be a function of class C^1 defined on the open set $U \subset R^{n-1}$. Let $S = \{(x, f(x)): x \in K\}$ where $K \subset U$ is a fixed compact set. It will suffice to prove there exists an $r > 0$ such that $v(B(y, r) \cap S, y)$ can be made arbitrarily small for all $y \in S$. Fix $y_0 = (x_0, f(x_0)) \in S$. Then for $r > 0$,

$$\begin{aligned} v(B(y_0, r) \cap S, y) &= \int_S n(y) \cdot \frac{(y - y_0)}{|y - y_0|^n} dH^{n-1}(y) \\ &= \int_K \frac{(\nabla f(x), 1)}{(|\nabla f(x)|^2 + 1)^{1/2}} \cdot \frac{(x - x_0, f(x) - f(x_0))}{|y - y_0|^n} \\ &\quad \times (|\nabla f(x)|^2 + 1)^{1/2} dL_{n-1}(x) \\ &= \int_K \frac{|\nabla f(x) \cdot (x - x_0) - [f(x) - f(x_0)]|}{|y - y_0|^n} dL_{n-1}(x)\ . \end{aligned}$$

Let $\rho = x - x_0$ and let points on the unit sphere in R^{n-1} be denoted by θ. Observe that there is a constant M such that

$$|x - x_0| \leqq |y - y_0| \leqq M|x - x_0|$$

for all $x_0, x \in K$. Then, with $\rho\theta = x - x_0$,

$$v(B(y_0, r) \cap S, y_0) \leqq N \iint ||\nabla f(x)| - |\nabla f(x_\theta)|| \frac{d\rho}{\rho} d\theta$$

where N is a constant and x_θ is some point on interval between x_0 and x. Hence, if the modulus of continuity of $|\nabla f|$, ω, is required to satisfy

$$\int_0^1 \frac{\omega(r)}{r} dr < \infty\ ,$$

then the conclusion obviously follows.

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