PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 2, 1974

LINEAR TRANSFORMATIONS ON SYMMETRIC SPACES

M. H. LIM

Let U be an n-dimensional vector space over an algebraically closed field F of characteristic zero, and let $\vee^r U$ denote the rth symmetric product space of U. Let T be a linear transformation on $\vee^r U$ which sends nonzero decomposable elements to nonzero decomposable elements. We prove the following:

(i) If $n = r + 1$ then T is induced by a nonsingular transformation on T.

(ii) If $2 < n < r + 1$ then either T is induced by a nonsingular transformation on U or $T(\vee^r U) = \vee^r W$ for some two dimensional subspace W of U.

The result for $n > r + 1$ was recently obtained by L. J. Cummings.

1. Preliminaries. Let U be a finite dimensional vector space over an algebraically closed field F. Let $\bigvee^r U$ denote the rth symmetric product space over U where $r \geqq 2$. Unlese otherwise stated, the characteristic of F is assumed to be zero or greater than r.

A *decomposable subspace* of $\bigvee^r U$ is a subspace consisting of decomposable elements. Let $x_1, \cdots, x_{r-1}$ be $r-1$ nonzero vectors in U. Then the set $\{x_1 \vee \cdots \vee x_{r-1} \vee u: u \in U\}$, denoted by $x_1 \vee \cdots \vee x_{r-1} \vee U$, is a decomposable subspace of $\bigvee^r U$ and is called a *type* 1 *subspace* of $\bigvee^r U$. Let W be a two dimensional subspace of U. It is shown in [2] that $\bigvee^r W$ is decomposable and is called a *type r subspace* of $\bigvee^r U$. If $y_1, \cdots, y_{r-k}$ are vectors in $U - W$ where $1 < k < r$, then the set $\{y_1 \vee \cdots \vee y_{r-k} \vee w_1 \vee \cdots \vee w_k: w_i \in W, i = 1, \cdots, k\}$, denoted by $y_1 \vee \cdots \vee y_{r-k} \vee W \vee \cdots \vee W$, is also decomposable and is called a *type k subspace* of $\bigvee^r U$. In [2] Cummings showed that every maximal decomposable subspce of $\bigvee^r U$ is of type i for some $1 \leqq i \leqq r$.

A linear transformation on $\bigvee^r U$ is called a *decomposable mapping* if it maps nonzero decomposable elements to nonzero decomposable elements. In [3] Cummings proved that if $\dim U > r + 1$ then every decomposable mapping T on $\bigvee^r U$ is induced by a nonsingular linear transformation f on U; that is, $T(y_1 \vee \cdots \vee y_r) = f(y_1) \vee \cdots \vee f(y_r)$. In this paper we consider the case when $3 \leqq \dim U \leqq r + 1$.

2. The case when $\dim U = r + 1$. Two type 1 subspaces M_1 and M_2 of $\bigvee^r U$ are called *adjacent* if

$$M_1 = x_1 \vee \cdots \vee x_{r-2} \vee y_1 \vee U$$
$$M_2 = x_1 \vee \cdots \vee x_{r-2} \vee y_2 \vee U$$

for some $x_1, \cdots, x_{r-2}, y_1, y_2$ where y_1 and y_2 are linearly independent.

The proof of the following lemma is contained in that of Proposition 4 of [3].

LEMMA 1. *The images of two adjacent type* 1 *subspaces under a decomposable mapping are distinct.*

THEOREM 1. *If* $\dim U = r + 1$ *then every decomposable mapping* T *of* $\bigvee^r U$ *is induced by a nonsingular mapping of* U.

Proof. Let M be a type 1 subspace of $\bigvee^r U$. Then $T(M)$ is a decomposable subspace of $\bigvee^r U$. Moreover $\dim M = \dim T(M) = r + 1$. Let $T(M) \subseteqq N$ where N is a maximal decomposable subspace. If N is of type k where $1 < k < r$, then $\dim N = k + 1 < r + 1$ which is a contradiction. Hence N is of type 1 or type r. Since $\dim N = r + 1$, it follows that $T(M) = N$.

Suppose that some type 1 subspace $x_1 \vee \cdots \vee x_{r-2} \vee y \vee U$ is mapped onto a type r subspace $\bigvee^r W$ where W is a two dimensional subspace of U. We shall show that this leads to a contradiction.

Let $\mathscr{C} = \{T(M_u): u \in U, u \neq 0\}$ where $M_u = x_1 \vee \cdots \vee x_{r-2} \vee u \vee U$. We shall show that $\bigvee^r W$ is the only type r subspace in $\mathscr{C}$. Suppose there is another type r subspace $\bigvee^r W^*$ in $\mathscr{C}$. Since $\bigvee^r W \cap \bigvee^r W^* \neq 0$, $W \cap W^*$ is 1-dimensional. Choose a nonzero vector z in U such that

$$T(x_1 \vee \cdots \vee x_{r-2} \vee y \vee z) = w_1 \vee \cdots \vee w_r$$

where $\dim \langle w_1, \cdots, w_r \rangle = 2$, $\langle y \rangle \neq \langle z \rangle$, and $W \cap W^* \neq \langle w_i \rangle$ for all $i = 1, \cdots, r$. If

$$T(M_z) = z_1 \vee \cdots \vee z_{r-1} \vee U$$

for some z_i in U then

$$T(M_z) \cap \bigvee^r W \neq 0$$

and

$$T(M_z) \cap \bigvee^r W^* \neq 0$$

imply that $z_1, \cdots, z_{r-1} \in W \cap W^*$ and hence $\langle z_1 \rangle = \cdots = \langle z_{r-1} \rangle = W \cap W^*$. Since $w_1 \vee \cdots \vee w_r \in z_1 \vee \cdots \vee z_{r-1} \vee U$, it follows that $\langle w_i \rangle = W \cap W^*$ for some i, a contradiction. Hence

$$T(M_z) = \bigvee^r S$$

for some two dimensional subspace S of U. Note that $x_1 \vee \cdots \vee x_{r-2} \vee y \vee z \in M_z \cap M_y$. Thus $w_1, \cdots, w_r \in W \cap S$. This implies that $\langle w_1, \cdots, w_r \rangle = W = S$, a contradiction to Lemma 1 since M_z and M_y

are adjacent type 1 subspaces. This proves that $\bigvee^r W$ is the only type r subspace in $\mathscr{C}$.

Since $\{T(M_x): \langle x\rangle \neq \langle y\rangle, x \neq 0\}$ is an infinite family of type 1 subspaces (Lemma 1) it follows from Proposition 4 of [3] that there exist vectors $u_1, \cdots, u_{r-2}$ such that for any $x \in U - \{0\}$ and $\langle x\rangle \neq \langle y\rangle$,

$$T(M_x) = u_1 \vee \cdots \vee u_{r-2} \vee x' \vee U$$

for some $x' \in U$. Since $T(M_x) \cap \bigvee^r W \neq 0$ we have $x' \in W$. Let g be a fixed nonzero vector such that $\langle g\rangle \neq \langle y\rangle$. Then for any $x \in U - \{0\}$ such that $\langle x\rangle \neq \langle g\rangle$, $\langle x\rangle \neq \langle y\rangle$,

$$T(x_1 \vee \cdots \vee x_{r-2} \vee x \vee g) = u_1 \vee \cdots \vee u_{r-2} \vee x' \vee g_x$$

for some g_x. Since $u_1 \vee \cdots \vee u_{r-2} \vee x' \vee g_x \in u_1 \vee \cdots \vee u_{r-2} \vee g' \vee U$ and $\langle x'\rangle \neq \langle g'\rangle$ we have $\langle g_x\rangle = \langle g'\rangle$. Therefore

$$\begin{aligned} T(M_g) \subseteqq u_1 \vee \cdots \vee u_{r-2} \vee g' \vee W \\ \cup \langle T(x_1 \vee \cdots \vee x_{r-2} \vee g \vee y)\rangle \\ \cup \langle T(x_1 \vee \cdots \vee x_{r-2} \vee g \vee g)\rangle\,. \end{aligned}$$

This is impossible since $\dim T(M_g) = \dim U > 2$.

Therefore, T maps type 1 subspaces to type 1 subspaces. By Theorem 2 of [3] T is induced by a nonsingular linear transformation on U.

3. The case when $3 \leqq \dim U < r + 1$. In this section we assume that char $F = 0$.

LEMMA 2. *Let $x_1, \cdots, x_k$ be k nonzero vectors of U. Let $r > k + 1$ and $x_1 \vee \cdots \vee x_k \vee A = z_1 \vee \cdots \vee z_r \neq 0$ in $\bigvee^r U$ where $A \in \bigvee^{r-k} U$ and $z_i \in U$. Then $\langle x_i\rangle = \langle z_{j_i}\rangle$ for some j_i where $j_s \neq j_t$ for distinct s and t.*

Proof. Let $u_1, \cdots, u_n$ be a basis of U. Let ϕ be the isomorphism from the symmetric algebra $\bigvee U$ over U onto the polynomical algebra $F[\xi_1, \cdots \xi_n]$ in n indeterminates $\xi_1, \cdots \xi_n$ over F such that $\phi(u_i) = \xi_i$, $i = 1, \cdots, n$ [4, p. 428]. Then

$$\phi(x_1) \cdots \phi(x_k)\phi(A) = \phi(z_1) \cdots \phi(z_r) \neq 0\,.$$

Since $F[\xi_1, \cdots, \xi_n]$ is a Gaussian domain and since $\phi(x_1), \cdots, \phi(x_k)$, $\phi(z_1), \cdots, \phi(z_r)$ are linear homogeneous polynomials, it follows that for each $i = 1, \cdots, k$, $\langle\phi(x_i)\rangle = \langle\phi(z_{j_i})\rangle$ for some j_i where $j_t \neq j_s$ if $s \neq t$. This implies that $\langle x_i\rangle = \langle z_{j_i}\rangle$. Hence the lemma is proved.

The following result is proved in [1, p. 131] under the assumption that char $F = 0$.

LEMMA 3. *$\bigvee^r U$ is spanned by* $\{u^r = \underbrace{u \vee \cdots \vee u}_{r\text{-times}} : u \in U\}$.

Hereafter we will assume that $3 \leqq \dim U < r + 1$ and T is a decomposable mapping on $\bigvee^r U$. Since every type k subspace has dimension $< r + 1$ where $1 \leqq k < r$ we see that every type r subspace of $\bigvee^r U$ is mapped onto a type r subspace under T.

LEMMA 4. *If there are two distinct type r subspaces M and N of $\bigvee^r U$ such that $M \cap N \neq 0$ and $T(M) = T(N)$, then $T(\bigvee^r U) = T(M)$.*

Proof. Let $M = \bigvee^r S_1$, $N = \bigvee^r S_2$ and $T(M) = T(N) = \bigvee^r S$ where S, S_1, S_2 are two dimensional subspaces of U. By hypothesis,

$$M \cap N = \bigvee^r S_1 \cap \bigvee^r S_2 = \bigvee^r (S_1 \cap S_2) \neq 0 .$$

Hence $S_1 \cap S_2$ is one dimensional. Let $S_1 = \langle y_1, y_2 \rangle$, $S_2 = \langle y_1, y_3 \rangle$. Consider $S_3 = \langle y_2, y_3 \rangle$. Then

$$\bigvee^r S_3 \cap \bigvee^r S_2 = \langle y_3^r \rangle , \qquad \bigvee^r S_3 \cap \bigvee^r S_1 = \langle y_2^r \rangle .$$

Hence $T(\bigvee^r S_3) \cap \bigvee^r S \supseteqq \langle T(y_3^r), T(y_2^r) \rangle$. Since T is a decomposable mapping and $\langle y_2^r, y_3^r \rangle$ is a two dimensional decomposable subspace, it follows that $\langle T(y_2^r), T(y_3^r) \rangle$ is two dimensional. Hence $T(\bigvee^r S_3) = \bigvee^r S$ because any two distinct type r subspaces of $\bigvee^r U$ have at most one dimension in common.

Let $z = \alpha y_1 + \beta y_2 + \gamma y_3$ where α, β, γ are all nonzero scalars. Consider $S_4 = \langle y_1, z \rangle = \langle y_1, \beta y_2 + \gamma y_3 \rangle$. Since

$$\bigvee^r S_4 \cap \bigvee^r S_3 \supseteqq \langle (\beta y_2 + \gamma y_3)^r \rangle ,$$
$$\bigvee^r S_4 \cap \bigvee^r S_1 \supseteqq \langle y_1^r \rangle ,$$

we have $T(\bigvee^r S_4) \cap \bigvee^r S \supseteqq \langle T(y_1^r), T((\beta y_2 + \gamma y_3)^r) \rangle$ which is two dimensional. Hence $T(\bigvee^r S_4) = \bigvee^r S$. Consequently by Lemma 3, $T(\bigvee^r \langle y_1, y_2, y_3 \rangle) = \bigvee^r S$.

Now, let $w \in U$ such that $w \notin \langle y_1, y_2, y_3 \rangle$. Let $W = \langle y_1, w \rangle$. Consider the type 1 subspace $P = y_1 \vee \cdots \vee y_1 \vee U$. Since

$$\dim (P \cap \bigvee^r \langle y_1, y_2, y_3 \rangle) = 3 ,$$

we have $\dim (T(P) \cap \bigvee^r S) \geqq 3$. Since the maximal dimension of the intersection of two distinct maximal decomposable subspaces is 2, we conclude that $T(P) \subseteqq \bigvee^r S$. This shows that

$$T(\bigvee^r W) \cap \bigvee^r S \supseteqq \langle T(y_1^r), T(y_1 \vee \cdots \vee y_1 \vee w) \rangle .$$

Since $\langle y_1^r, y_1^{r-1} \vee w \rangle$ is a two dimensional decomposable subspace, $\langle T(y_1^r), T(y_1^{r-1} \vee w) \rangle$ is also two dimensional. Hence $T(\bigvee^r W) = \bigvee^r S$. By Lemma 3, we conclude that $T(\bigvee^r U) = \bigvee^r S$. This completes the proof.

LEMMA 5. *Suppose that for any two distinct type r subspaces M, N such that $M \cap N \neq 0$, we have $T(M) \neq T(N)$. Then T is induced by a nonsingular transformation on U.*

Proof. Let y, y_1, y_2 be linearly independent vectors. Let $S_1 = \langle y, y_1\rangle$, $S_2 = \langle y, y_2\rangle$. Then $T(\mathsf{V}^r S_1) = \mathsf{V}^r S_1'$ and $T(\mathsf{V}^r S_2) = \mathsf{V}^r S_2'$ for some two dimentional subspaces S_1', S_2' of U. By hypothesis $\mathsf{V}^r S_1' \neq \mathsf{V}^r S_2'$. Hence

$$\mathsf{V}^r S_1' \cap \mathsf{V}^r S_2' = T(\mathsf{V}^r S_1 \cap \mathsf{V}^r S_2) = \langle y'^r\rangle$$

for some $y' \in U$. Therefore $T(y^r) = \lambda y'^r$ for some λ in F.

Let $H = y \vee \cdots \vee y \vee U$. We claim that $T(H) = y' \vee \cdots \vee y' \vee U$. Since $T(H)$ is a decomposable subspace, it is contained in a maximal decomposable subspace. If $T(H)$ is contained in a type k subspace $g_1 \vee \cdots \vee g_{r-k} \vee W \vee \cdots \vee W$ where $2 \leqq k < r$, then $y'^r \in g_1 \vee \cdots \vee g_{r-k} \vee W \vee \cdots \vee W$ and hence $\langle g_1\rangle = \langle y'\rangle$, $y' \in W$. This implies $g_1 \in W$, a contradiction. If $T(H)$ is contained in a type r subspace $\mathsf{V}^r W$, then

$$\dim(\mathsf{V}^r S_1 \cap H) = 2 \Longrightarrow \dim(T(\mathsf{V}^r S_1) \cap \mathsf{V}^r W) \geqq 2\,,$$
$$\dim(\mathsf{V}^r S_2 \cap H) = 2 \Longrightarrow \dim(T(\mathsf{V}^r S_2) \cap \mathsf{V}^r W) \geqq 2\,.$$

Since $T(\mathsf{V}^r S_1)$ and $T(\mathsf{V}^r S_2)$ are both type r subspaces, it follows that $T(\mathsf{V}^r S_1) = \mathsf{V}^r W = T(\mathsf{V}^r S_2)$, a contradiction to our hypothesis. Hence $T(H)$ is a type 1 subspace of $\mathsf{V}^r U$. Since $y'^r \in T(H)$, it follows that

$$T(H) = y' \vee \cdots \vee y' \vee U\,.$$

By Lemma 3, let $x_1^{r-1}, \cdots, x_t^{r-1}$ be a basis of $\mathsf{V}^{r-1} U$. Note that $3 \leqq \dim U < r + 1$ implies that $r \geqq 3$. Clearly if $i \neq j$ then x_i and x_j are linearly independent. Consider any type one subspace $D = z_1 \vee \cdots \vee z_{r-1} \vee U$. Let $z_1 \vee \cdots \vee z_{r-1} = \sum_{i=1}^t \lambda_i x_i^{r-1}$ where $\lambda_i \in F$ and $i = 1, \cdots, t$. We shall show that $T(D)$ is a type 1 subspace. Suppose to the contrary that

(i) $T(D) \subsetneqq \mathsf{V}^r S$

or

(ii) $T(D) \subsetneqq w_1 \vee \cdots \vee w_{r-k} \vee S \vee \cdots \vee S$, $2 \leqq k < r$,

for some two dimensional subspace S of U and some $w_1, \cdots, w_{r-k} \in U - S$.

Let $T(x_i \vee \cdots \vee x_i \vee U) = x_i' \vee \cdots \vee x_i' \vee U$, $i = 1, \cdots, t$. Note that $T(x_i^r) = \eta_i x_i'^r$ for some $\eta_i \in F$, $i = 1, \cdots, t$. For $i \neq j$, $\langle x_i^r, x_j^r\rangle$ is a two dimensional subspace of $\mathsf{V}^r U$ implies that $T(\langle x_i^r, x_j^r\rangle) = \langle x_i'^r, x_j'^r\rangle$ is a two dimensional subspace of $\mathsf{V}^r U$. Hence x_i' and x_j' are linearly independent if $i \neq j$.

Consider case (ii). Choose a vector w of U such that

$$w \notin \langle w_1 \rangle \cup \cdots \cup \langle w_{r-k} \rangle \cup S \cup \Big(\bigcup_{i \neq j} \langle x'_i, x'_j \rangle \Big) .$$

Let $u \in U$ such that $T(x_1^{r-1} \vee u) = x_1'^{r-1} \vee w$. For each $i \geqq 2$, let $T(x_i^{r-1} \vee u) = x_i'^{r-1} \vee u_i$. We shall show that $\langle u_i \rangle = \langle w \rangle$ for $i \geqq 2$.

Since $\langle x_{1}^{r-1} \vee u, x_i^{r-1} \vee u \rangle$ is a decomposable subspace for $i \geqq 2$, $\langle x_1'^{r-1} \vee w, x_i'^{r-1} \vee u_i \rangle$ is also a decomposable subspace. By our choice of w, $\langle x'_1, w, x'_i \rangle$ is three dimensional. Hence $\langle x_1'^{r-1} \vee w, x_i'^{r-1} \vee u_i \rangle$ is contained in a type k subspace A for some $1 \leqq k < r$. If A is of type k where $1 \leqq k \leqq r-2$, then we have $\langle x'_i \rangle = \langle w \rangle$ or $\langle x'_i \rangle = \langle x'_1 \rangle$ which is a contradiction. Hence A is of type $r-1$. This implies that $\langle u_i \rangle = \langle w \rangle$, $i \geqq 2$.

Let $u_i = a_i w$ where $a_i \in F$, $i \geqq 2$. Then

$$\begin{aligned} T(z_1 \vee \cdots \vee z_{r-1} \vee u) &= T\Big(\sum_{i=1}^{t} \lambda_i x_i^{r-1} \vee u\Big) \\ &= \lambda_1 x_1'^{r-1} \vee w + \sum_{i=2}^{t} \lambda_i x_i'^{r-1} \vee (a_i w) \\ &= \Big(\lambda_1 x_1'^{r-1} + \sum_{i=2}^{t} \lambda_i a_i x_i'^{r-1}\Big) \vee w \\ &= g_1 \vee \cdots \vee g_r \neq 0 \end{aligned}$$

for some $g_i \in U$, $i = 1, \cdots, r$. In view of Lemma 2, $\langle g_j \rangle = \langle w \rangle$ for some j, $1 \leqq j \leqq r$. Since

$$g_1 \vee \cdots \vee g_r \in w_1 \vee \cdots \vee w_{r-k} \vee S \vee \cdots \vee S ,$$

we have $\langle w \rangle = \langle w_i \rangle$ for some i or $w \in S$. This contradicts our choice of w. Hence

$$T(D) \nsubseteq w_1 \vee \cdots \vee w_{r-k} \vee S \vee \cdots \vee S .$$

Similarly $T(D) \nsubseteq \bigvee^r S$. Therefore $T(D)$ is a type 1 subspace. In view of Theorem 2 of [3], T is induced by a nonsingular linear transformation on U.

Combining Lemmas 4 and 5 we have the following main result:

THEOREM 2. *Let* $T\colon \bigvee^r U \rightarrow \bigvee^r U$ *be a decomposable mapping. If* $3 \leqq \dim U < r+1$ *then either* T *is induced by a nonsingular transformation on* U *or* $T(\bigvee^r U)$ *is a type* r *subspace. In particular, if* T *is nonsingular, then* T *is induced by a nonsingular transformation on* U.

We have so far not been able to determine whether there does in fact exist a decomposable mapping on $\bigvee^r U$ such that its image is a type r subspace when $3 \leqq \dim U < r+1$.

The author is indebted to Professor R. Westwick for his encouragement and suggestions. Thanks are also due to the referee for his suggestions.

References

1. H. Boerner, *Representations of Groups*, North-Holland, Amsterdam, 1963.
2. L. J. Cummings, *Decomposable symmetric tensors*, Pacific J. Math., **35** (1970), 65-77.
3. ———, *Transformations of symmetric tensors*, Pacific J. Math., **42** (1972), 603-613.
4. S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965.

Received October 3, 1973 and in revised form September 18, 1974.

UNIVERSITY OF MALAYA, KUALA LUMPUR, MALAYSIA