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PRERADICALS AND INJECTIVITY

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Recently, the (ρ, σ)-injectivity of modules with respect to a couple of preradicals has been investigated. In the general case, the study of all the (ρ, σ)-injectivities reduces to that with σ a torsion preradical. For a special class of rings the (ρ, σ)-injectivities are completely described. The description of all quasi-injective modules over a Dedekind domain appears as a simple corollary.

J. A. Beachy [1] has introduced a new concept of ρ-density of a submodule N of a module M and he has investigated the (ρ, σ)-injectivity of modules with respect to a couple of preradicals. In this paper we shall show that to any couple (ρ, σ) of preradicals there exists a torsion preradical σ' such that the (ρ, σ)-injectivity and (ρ, σ')-injectivity have the same meanings. Further, in the study of (ρ, σ)-injectivity, where ρ is a torsion preradical, ρ can be replaced by a torsion radical. Finally, the (ρ, σ)-injectivities are completely determined for a class of subcommutative rings (containing all Dedekind domains) and this yields a characterization of all quasi-injective modules generalizing a result of Harada [7] (the methods are quite different).

We start with some basic definitions. A preradical ρ for the category ${}_R\mathcal{M}$ of left R-modules over an associative ring R with unity is any subfunctor of the identity, i.e. ρ assigns to each module M a submodule $\rho(M)$ in such a way that every homomorphism $M \to N$ induces $\rho(M) \to \rho(N)$ by restriction. A preradical ρ is said to be idempotent if $\rho^2 = \rho$, torsion if ρ is left exact and it is called a radical if $\rho(M/\rho(M)) = 0$. It is well-known that ρ is torsion iff $L \subseteq M$ implies $\rho(L) = L \cap \rho(M)$ (see e.g. [10], Prop. 1.4). For a preradical ρ, a module M is called ρ-torsion if $\rho(M) = M$ and ρ-torsion-free if $\rho(M) = 0$. Following J. A. Beachy [1] a submodule N of a module M is called ρ-dense in M if $M/N \subseteq \rho(K/N)$ for some module K containing M, or, equivalently, if $M/N \subseteq \rho(\hat{M}/N)$ where $\hat{M}$ denotes the injective hull of M. Finally, for a couple (ρ, σ) of preradicals a module Q is said to be (ρ, σ)-injective if for every diagram $f \downarrow \begin{smallmatrix} N_0 \to N \\ Q \end{smallmatrix}$ with N_0 ρ-dense in N and Ker f σ-dense in N there is $g : N \to Q$ making this diagram commutative. If ρ is a preradical and M a module then the module Q is said to be (ρ, M)-injective if every diagram $f \downarrow \begin{smallmatrix} M_0 \to M \\ Q \end{smallmatrix}$ with M_0 ρ-dense in M can be made commutative by some homomorphism $M \to Q$.

For a preradical σ let σ' be the smallest torsion preradical which contain σ. Then $\sigma'(M) = M \cap \sigma(\hat{M})$, so $\sigma'(M) = \sigma(M)$ if M is injective, and M is σ-torsion iff 0 is σ-dense in M. For a preradical ρ one can construct an ordinal sequence of preradicals in the following way:

$$\rho'(A) = \rho(A),$$

$$\rho^{\alpha+1}(A)/\rho^{\alpha}(A) = \rho(A/\rho^{\alpha}(A)),$$

$$\rho^{\alpha}(A) = \bigcup_{\beta<\alpha} \rho^{\beta}(A), \ \alpha \text{ a limit ordinal.}$$

As it is well-known (see **[10]**), the preradical ρ^* defined by $\rho^*(A) = \rho^{\alpha}(A)$ whenever $\rho^{\alpha}(A) = \rho^{\alpha+1}(A)$ is a radical and, in fact, the smallest radical containing ρ (we put $\rho \leqq \sigma$ whenever $\rho(M) \subseteq \sigma(M)$ for all modules M).

LEMMA 1. *Let ρ, σ be preradicals for ${}_R\mathcal{M}$. A module Q is (ρ, σ)-injective iff it is (ρ, M)-injective for all modules M having 0 as a σ-dense submodule.*

Proof. See **[1]**, Theorem 23.

THEOREM 2. *Let ρ, σ be preradicals for ${}_R\mathcal{M}$ and let $Q \in {}_R\mathcal{M}$. Then*

(a) *Q is (ρ, σ)-injective iff it is (ρ, σ')-injective,*

(b) *if ρ is a torsion preradical, then Q is (ρ, σ)-injective iff it is (ρ^*, σ)-injective.*

Proof. For to prove (a) it suffices to use Lemma 1, since 0 is σ-dense in M iff 0 is σ'-dense in M.

If Q is (ρ^*, σ)-injective then it is (ρ, σ)-injective since $\rho \leqq \rho^*$. Assume that Q is (ρ, σ)-injective, and let $N_0 \subseteq N$ be ρ^*-dense, with $f: N_0 \to Q$ and Ker f σ-dense in N. By a well-known argument using Zorn's lemma there exists a maximal extension $f_1: N_1 \to Q$. Then $\rho(N/N_1) = 0$, since otherwise f_1 could be extended to the ρ-closure of N_1 in N (by (a), σ can be assumed to be a torsion preradical), and so therefore $\rho^*(N/N_1) = \rho(N/N_1) = 0$, which implies $N_1 = N$.

THEOREM 3. *If R is left hereditary, then the following equivalent conditions hold for each preradical ρ for ${}_R\mathcal{M}$.*

(1) *$M_0 \subseteq M$ is ρ-dense iff it is ρ'-dense,*

(2) *if $M_0 \subseteq M$ is ρ-dense and $M/M_0 \cong N/N_0$, then $N_0 \subseteq N$ is ρ-dense.*

Proof. The equivalence of conditions (1) and (2) is obvious. If R is left hereditary, then if $M_0 \subseteq M$, $\hat{M}/M_0$ is injective, and so M_0 is ρ-dense in M iff $M/M_0 \subseteq \rho(\hat{M}/M_0) = \rho'(\hat{M}/M_0)$ iff M_0 is ρ'-dense in M.

Before proceeding we recall some basic definitions (see e.g. **[2]**). Let π be the set of all pair-wise non-isomorphic simple left R-modules. For every module M and every subset $\pi' \subseteq \pi$ let us define $S_{\pi'}(M)$ as the submodule of M generated by all simple submodules of M isomorphic to some module from π'. It is easy to see that $S_{r'}$ is a torsion preradical. The smallest radical $S^*_{\pi'}$ containing $S_{\pi'}$ (defined above) is torsion and is said to be the fundamental torsion radical. A ring R is said to have primary decompositions (PD) if $S^*_{\pi}(M) = \Sigma^{\circ}_{U\in\pi} S^*_{U}(M)$ for every module M. It is well-known that for a subcommutative ring with (PD) for which M/M^2 is either 0 or a simple module for every maximal ideal M there is $S^*_{\pi} = S^{\omega}_{\pi}$ where ω is the first infinite ordinal (see e.g. **[9]**). Recall (**[9]**, Def. 7.1) that a S^*_{π}-torsion module M is said to be quasicyclic if $S^{\alpha+1}_{\pi}(M)/S^{\alpha}_{\pi}(M)$ is either 0 or simple for all ordinals α and $S^{n}_{\pi}(M) \subsetneq M$ for all natural integers n. For further purposes we denote by 0 the zero functor and by ∞ the identity functor. In the rest of this paper we shall deal with a subcommutative ring R having (PD) such that M/M^2 is either 0 or a simple module for every maximal ideal M, every proper homomorphic image of R is S^*_{π}-torsion and every preradical for ${}_R\mathcal{M}$ satisfies condition (2) of Theorem 3. For easy references we shall call such a ring a BS-ring. The last condition is independent from all others as shows the following example: Taking as $R = Z/(p^3)$ the factor-ring of integers modulo p^3 and $\rho(M) = JM$ where J is the Jacobson radical of R, we obtain a preradical which does not satisfy the condition (2) from Theorem 3, since $J \cdot C(p^3) = C(p)$ so that 0 is ρ-dense in $C(p^2)$. On the other hand, $C(p^3)/C(p^2)$ is not ρ-torsion, so that $C(p)$ is not ρ-dense in $C(p^3)$, $C(p^3)$ being injective. The idempotent radical σ on the abelian groups category assigning to each group its greatest divisible subgroup provides an example of a preradical which is not torsion and satisfies the condition (2) from Theorem 3.

THEOREM 4. *Let R be a BS-ring. Then $\rho \neq \infty$ is a torsion preradical for ${}_R\mathcal{M}$ iff to every $U \in \pi$ there is $n(U) \in \{N \cup \{0\} \cup \{\infty\}$, N the set of natural integers$\}$ such that $\rho(M) = \Sigma^{\circ}_{U\in\pi} S^{n(U)}_{U}(M)$ for all $M \in {}_R\mathcal{M}$.*

Proof. We can obviously restrict ourselves to the proof of the necessity. It is well known that the smallest radical ρ^* containing ρ is torsion. Now from the correspondence between torsion radicals and radical filters (see e.g. **[8]**) and from the fact every proper homomorphic image of R is S^*_{π}-torsion it easily follows ρ^* is fundamental, $\rho^* = S^*_{\pi'}$

for some $\pi' \subseteq \pi$. Thus for every $M \in {}_R\mathcal{M}$, $\rho(M) = \Sigma^{\circ}_{U \in \pi} M_U = \Sigma^{\circ}_{U \in \pi} \rho_U(M)$ where $\rho_U = S^*_U \rho$. Now it suffices to describe ρ_U. Let I be a maximal ideal of R such that $R/I \cong U$. Two cases can arise:

(1) All the cyclic modules R/I^n, $n = 1, 2, \cdots$ are ρ-torsion. If $I^n = I^{n+1}$ for some natural integer n, then $S^n_U = S^{n+1}_U$ and $\rho_U = S^n_U = S^*_U$. If $I^n \underset{\neq}{\supset} I^{n+1}$ for all n then as in the case of abelian groups the U-quasicyclic module (i.e. quasicyclic module M with $S^{n+1}_\pi(M)/S^n_\pi(M) \cong U$) is a direct limit of R/I^n, $n = 1, 2, \cdots$ and thus $\rho_U = S^*_U$, since by **[2]**, Theorem 3.3 every S^*_U-torsion module can be embedded in a direct sum of quasicyclic modules.

(2) There exists a nonnegative integer n such that R/I^n is ρ-torsion and R/I^k, $k > n$ is not ρ-torsion. Now by **[2]**, Theorem 4.2 for every $M \in {}_R\mathcal{M}$ $S^n_U(M)$ is a direct sum of cyclic submodules each of which is isomorphic to some R/I^l, $l \leqq n$ and hence $S^n_U \leqq \rho_U$. On the other hand, for any $M \in {}_R\mathcal{M}$ every cyclic submodule of $\rho_U(M)$ is isomorphic to some R/I^l, $l \leqq n$, so that $I^n \rho_U(M) = 0$ and $S^n_U = \rho_U$.

THEOREM 5. *Let R be a BS-ring and M a module. Then the following hold*:

(i) *If M is not S^*_π-torsion then a module Q is M-injective iff it is injective,*

(ii) *if M is S^*_π-torsion then a module Q is M-injective iff $S^{n(U)}_U(Q) = S^{n(U)}_U(\hat{Q})$ for all $U \in \pi$, where $n(U)$ is the smallest ordinal for which $S^{n(U)}_U(M) = S^{n(U)+1}_U(M)$.*

Proof. By **[1]**, Corollary 2.9 Q is M-injective iff it is (∞, ρ)-injective where ρ is the smallest torsion preradical for which M is torsion.

(i) Taking an element $x \in M - S^*_\pi(M)$ we have $Rx \cong R$, R being a BS-ring. Thus $\rho(R) = R$ and $\rho = \infty$. It is now obvious that Q is (∞, ∞)-injective iff it is injective.

(ii) By Theorem 4, $\rho(N) = \Sigma^{\circ}_{U \in \pi} S^{n(U)}_U(N)$ for every $N \in {}_R\mathcal{M}$. As it is easily seen the numbers $n(U)$ are just the smallest ordinals for which $S^{n(U)}_U(M) = S^{n(U)+1}_U(M)$. By **[1]**, Theorem 2.5 a module Q is $(\infty, S^{n(U)}_U)$-injective iff $S^{n(U)}_U(Q) = S^{n(U)}_U(\hat{Q})$ and the assertion follows.

COROLLARY 6. *Let R be a BS-ring and Q a module. Then Q is quasi-injective iff it is either injective or of the form $Q = \Sigma^{\circ}_{U \in \pi} S^*_U(Q)$ where every $S^*_U(Q)$, $U \in \pi$ is a direct sum of pair-wise isomorphic cyclic or quasi-cyclic modules.*

Proof. We proceed to the necessity, the sufficiency being obvious by Theorem 5. Suppose that Q is quasi-injective. By the preceding

Theorem Q is either injective or of the form $Q = \Sigma^{\circ}_{U \in \pi} S^*_U(Q)$. By **[2]**, Theorems 3.2 and 4.2 every $S^*_U(Q)$ is a direct sum of quasicyclic and cyclic modules, so that it suffices to use Theorem 5 (ii).

THEOREM 7. *Let R be a BS-ring and $\rho, \sigma \neq \infty$ be two preradicals for ${}_R\mathcal{M}$. Then there exists a module M such that a module Q is (ρ, σ)-injective iff it is M-injective. Moreover, M can be chosen quasi-injective.*

Proof. We shall divide this proof into three steps.

(1) We show that if ρ, σ are torsion preradicals such that $\rho^* \cap \sigma$, $\sigma \neq \infty$ and Q is a (∞, ρ)-injective module, then Q is (ρ, σ)-injective. So, let Q be a (∞, ρ)-injective module and let us consider the diagram

$$\begin{array}{l} K \\ \downarrow \\ N_0 \rightarrow N \\ f \downarrow \\ Q \end{array}$$

with N/N_0 ρ-torsion, N/K σ-torsion, $K = \operatorname{Ker} f$. It follows from $\rho = \rho^* \cap \sigma$ and Theorem 4 that $\sigma(M) = \rho(M) \oplus \tau(M)$ for every $M \in {}_R\mathcal{M}$ and a suitable torsion preradical τ. For $N'/K = \tau(N/K)$ the module $(N' + N_0)/N_0 \cong N'/N_0 \cap N'$ is ρ-torsion and τ-torsion so that $N' \subseteq N_0$. Thus f induces

$$\begin{array}{l} N_0/K = \tau(N/K) \oplus (\rho(N/K) \cap N_0/K) \rightarrow \tau(N/K) \oplus \rho(N/K) \\ \bar{f} \downarrow \\ Q \end{array}$$

By hypothesis, the restriction of $\bar{f}$ to $\rho(N/K) \cap N_0/K$ extends to a homomorphism $\rho(N/K) \rightarrow Q$ and the assertion follows easily.

(2) It follows from Theorems 2 and 3 that Q is (ρ, σ)-injective iff it is $((\rho')^*, \sigma')$-injective. Now by the definition Q is $((\rho')^*, \sigma')$-injective iff it is $((\rho')^* \cap \sigma', \sigma')$-injective and the preceding part results that Q is (ρ, σ)-injective iff it is (∞, τ)-injective, where $\tau = (\rho')^* \cap \sigma'$.

(3) From (2) and Theorem 4 it easily follows that a module Q is (∞, τ)-injective iff it is $(\infty, S_U^{n(U)})$-injective for all $U \in \pi$ where $\tau(M) = \Sigma^{\circ}_{U \in \pi} S_U^{n(U)}(M)$. By **[1]**, Theorem 2.5 Q is $(\infty, S_U^{n(U)})$-injective iff $S_U^{n(U)}(\hat{Q}) \subseteq Q$ and the idempotence of $S_U^{n(U)}$ yields that Q is (ρ, σ)-injective iff $S_U^{n(U)}(Q) = S_U^{n(U)}(\hat{Q})$. For $U \in \pi$, $U \cong R/I, I$ a maximal ideal of R, we put $M_U = R/I^{n(U)}$ if $n(U) \in N \cup \{0\}$ and M_U is an

U-quasicyclic module if $n(U) = \infty$. Taking $M = \Sigma^{\circ}_{U \in \pi} M_U$ it suffices to use Theorem 5 (ii). M is quasi-injective by Corollary 6.

REMARK. M. Harada ([7], Corollary to Proposition 2.6) has described the structure of quasi-injective modules over a Dedekind domain. This description follows from our Corollary 6 immediately.

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