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ON ELEMENTARY IDEALS OF PROJECTIVE PLANES IN THE 4-SPHERE AND ORIENTED θ-CURVES IN THE 3-SPHERE

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The concept of an infinite cyclic covering has been applied to knot theory. In this paper that of a finite cyclic covering is considered. This enable us to study such cases as projective planes in the 4-sphere and oriented θ-curves in the 3-sphere. Some properties of elementary ideals of these cases are examined. The technique of free differential calculus is used, instead of that of coverings.

Let L be a polyhedron in an n-sphere S^n $(n > 1)$ that does not separate S^n, and let G_L be the fundamental group of $S^n - L$. We use the additive group J_p of integers modulo p as the coefficient group for homology. Let l be an $(n-2)$-dimensional cycle on L. Let H_p be the multiplicative cyclic group of order p, generated by t. Then, there is a homomorphism ψ of G_L into H_p such that for each $g \in G_L$,

$$g\psi = t^{\operatorname{link}(g,l)} ,$$

where $\operatorname{link}(g, l) \in J_p$ is the linking number between g and l in S^n.

Using Fox's free differential calculus ([1], [2]), we associate to ψ a sequence of elementary ideals $E_d(G_L, \psi)$ of the group G_L, evaluated in the group ring JH_p of H_p over integers J. This sequence of elementary ideals depends only on G_L and ψ, and hence it depends only on the position of l on L in S^n. We shall denote it by $E_d(l)$. If l and l' are homologous on L, then $E_d(l) = E_d(l')$ for every d.

In this paper we apply these elementary ideals $E_d(l)$ to the study of the position of L in S^n. The following two cases of $E_d(l)$ are considered: (1) L is a projective plane in S^4 and $p = 2$, and (2) L is a θ-curve in S^3 and $p = 3$.

1. Miscellanea.

Let $\sigma(t) = 1 + t + \cdots + t^{p-1} \in JH_p$.

THEOREM 1. *If ψ is onto, then $E_0(l) \subset (\sigma(t))$ in JH_p.*

Proof. It is proved in [2] that

$$E_0(H_p, id) = (\sigma(t)) ,$$

where id is the identity isomorphism of H_p. From the diagram

$$G_L \xrightarrow{\psi} H_p \xrightarrow{id} H_p ,$$

where ψ is onto, and Theorem 1 in [4], it follows that $E_0(l) \subset (\sigma(t))$ in JH_p.

Now assume that ψ is onto, and let $E_0(l) = \sigma(t)E(l)$. Let $\omega = e^{2\pi i/p}$ and let $J[\omega]$ be the ring of all complex numbers of the form $\sum_{i=0}^{p-1} a_i\omega^i$, where $a_i \in J$ $(i = 0, 1, \cdots, p-1)$. A homomorphism * of H_p into $J[\omega]$ is defined by $t^* = \omega$. We naturally extend * to a ring homomorphism of JH_p onto $J[\omega]$. Though $E_0(l)^* = (0)$, sometimes $E(l)^*$ is a nontrivial ideal in $J[\omega]$.

A trivializer of a group G is a homomorphism of G onto the trivial group that consists of only one element. Any trivializer will be denoted by the same notation $\circ$ in this paper. Further the group ring JG° will be identified with J.

2. Projective planes in S^4. Let P be a polyhedral projective plane in S^4. By the Alexander duality theorem, the abelianization of the fundamental group G_P of $S^4 - P$ is a cyclic group of order 2. We use J_2 as the coefficient group for homology. Let l be a 2-cycle on P.

THEOREM 2. $\begin{cases} E_0(l)^\circ = (2) \text{ and} \\ E_d(l)^\circ = (1), \text{ if } d > 0, \text{ in } J. \end{cases}$

Proof. This follows to Theorem 2 in [4].

A projective plane P has only two cycles. First let l_0 be the trivial one.

THEOREM 3. $\begin{cases} E_0(l_0) = (2) \text{ and} \\ E_d(l_0) = (1), \text{ if } d > 0, \text{ in } JH_2. \end{cases}$

Proof. The proof is similar to that of Theorem 3 in [4].

Now let l be the nontrivial 2-cycle on P, i.e., the fundamental cycle for J_2-orientation of P. Since the homomorphism ψ is onto in this case, by Theorem 1 we have $E_0(l) \subset (1 + t)$ in JH_2. Let $E_0(l) = (1 + t)E(l)$.

THEOREM 4. *$E(l)^\circ = (1)$ in J.*

Proof. Since

$$(2) = E_0(l)^\circ = (1 + t)^\circ E(l)^\circ = (2)E(l)^\circ$$

in J, we have $E(l)^\circ = (1)$ in J.

Further $E(l)^* \subset J$ is also an invariant of P in S^4.

THEOREM 5. *The ideal $E(l)^*$ in J is generated by an odd integer.*

Proof. Let $E(l)$ be generated by $a_i + b_i t$ $(i = 1, 2, \cdots, n)$ in JH_2. Assume on the contrary that $E(l)^*$ is not generated by an odd integer. Then we have $a_i - b_i = 0$ mod. 2 for every i. From this it follows that $a_i + b_i = 0$ mod. 2 for every i. Hence we have $E(l)^\circ \neq (1)$ in J which contradicts Theorem 4.

EXAMPLE 1. Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then, for each $f(t)$ there is a polyhedral, locally flat projective plane P_f in S^4, where the odd natural number $|f(-1)|$ is a topological invariant of P_f in S^4 (see [3]). In these example, it is easy to see that for the nontrivial 2-cycle l on P_f we have $E(l) = (f(t))$ in JH_2, where $f(t)$ is considered as an element of JH_2. Further we have $E(l)^* = (f(-1))$ in J.

3. θ-curves in S^3. Let P and Q be two distinct points in S^3 and let a_1, a_2, and a_3 be three polygonal arcs from P to Q, which are mutually disjoint to each other except at P and Q. Then $L = a_1 \cup a_2 \cup a_3$ is called a θ-curve in S^3. Further, if each of these three arcs is oriented from P to Q, then L is called an oriented θ-curve in S^3. From now on we use J_3 as the coefficient group for homology.

Let L be a θ-curve in S^3. Then the abelianization of the fundamental group of $S^3 - L$ is a free abelian group of rank 2. Let l be a 1-cycle on L.

THEOREM 6. $\begin{cases} E_0(l)^\circ = E_1(l)^\circ = (0) \text{ } and \\ E_d(l)^\circ = (1), \text{ } if \ d > 1, \text{ } in \ J. \end{cases}$

Proof. This follows to Theorem 9 in [4].

THEOREM 7. $E_0(l) = E_1(l) = (0)$ *in* JH_3.

Proof. This follows to corollary of Theorem 7 in [4].

Now let L be an oriented θ-curve in S^3. Then there is a nontrivial 1-cycle l on L such that the coefficient of l for each oriented 1-simplex of L is $1 \in J_3$. The 1-cycle l is called the fundamental cycle for the J_3-orientation of L. Then, $E_2(l)$ in JH_3 and $E_2(l)^*$ in $J[\omega]$, where $\omega = e^{2\pi i/3}$, are topological invariants of the oriented θ-curve L in S^3.

EXAMPLE 2. Let L be the example of an oriented θ-curve in [4], where the orientation of L is given as shown in the figure in [4]. Let l be the fundamental cycle for this J_3-orientation of L. Then we have

$$\begin{cases} E_0(l) = E_1(l) = (0)\,, \\ E_2(l) = (t^2 + t + 1,\, 2) \quad \text{and} \\ E_d(l) = (1)\,, \quad \text{if} \quad d > 2\,, \end{cases}$$

in JH_3 and $E_2(l)^* = (2)$ in $J[\omega]$.

THEOREM 8. *Let $f(t) \in JH_3$ with $f(1) = 1$. Then there is an oriented θ-curve L in S^3 such that for the fundamental cycle l for the J_3-orientation of L we have*

$$\begin{cases} E_0(l) = E_1(l) = (1)\,, \\ E_2(l) = (f(t)) \quad \textit{and} \\ E_d(l) = (1)\,, \quad \textit{if} \quad d > 2\,, \quad \textit{in} \quad JH_3\,. \end{cases}$$

Proof. Let $f(\tau) \in JH$ with $f(1) = 1$, where H is an infinite cyclic multiplicative group generated by τ. Then there is an example of a θ-curve L_1 and a 1-cycle l_1 on L_1 such that

$$\begin{cases} E_0(l_1) = E_1(l_1) = (1)\,, \\ E_2(l_1) = (f(\tau)) \quad \text{and} \\ E_d(l_1) = (1)\,, \quad \text{if} \quad d > 2\,, \end{cases}$$

in JH (see [5]). The coefficients of l_1 on L_1 are distributed as shown in Fig. 1. Note that arcs in the outside of the cube shown by dotted lines in the figure are possibly complicated. Now the sequence of elementary ideals remains invariant, even if L_1 is "blown up" to a cube with 2 handles. Then the 1-cycle l_2 on L_2 as shown in Fig. 4 has the same sequence of elementary ideals to that of l_1 on L_1. Considering l_2 in the homology of integers modulo 3, we have an example of a 1-cycle l on L as shown in Fig. 5. The 1-cycle l is the fundamental cycle of a J_3-orientation of the θ-curve L and for each d we have $E_d(l) = E_d(l_1)'$ in JH_3, where $'$ is a ring homomorphism

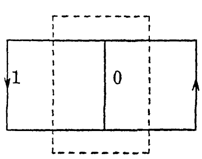

Fig. 1

$\Rightarrow$

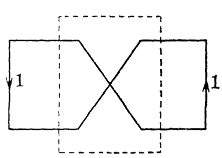

Fig. 2

$\Rightarrow$

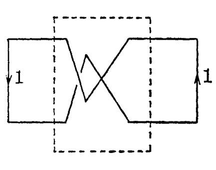

Fig. 3

$\Rightarrow$

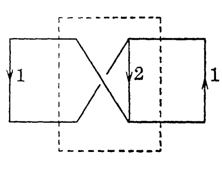

Fig. 4

$\Rightarrow$ mod. 3

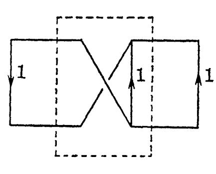

Fig. 5

of JH onto JH_3 defined by $\tau' = t$. Now the theorem can be seen easily.

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