PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No 1, 1975

SUBHARMONICITY AND HULLS

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For X a compact set in C^2, $h(X)$ denotes the polynomially convex hull of X. We are concerned with the existence of analytic varieties in $h(X)\setminus X$. X is called "invariant" if (z, w) in X implies $(e^{i\theta}z, e^{-i\theta}w)$ is in X, for all real θ. X is called an "invariant disk" if there is a continuous complex-valued function a defined on $0 \leqq r \leqq 1$ with $a(0) = a(1) = 0$, such that $X = \{(z, w) \mid |z| \leqq 1,\ w = a(|z|)/z\}$. Let X be an invariant set and put $f(z, w) = zw$. Let Ω be an open disk in $C\setminus f(X)$ and put $f^{-1}(\Omega) = \{(z, w)$ in $h(X) \mid zw \in \Omega\}$. In Theorem 2 we show that if $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic variety. Let now X be an invariant disk, with certain hypotheses on the function a. Then we show in Theorem 3 that $f^{-1}(\Omega)$ is the union of a one-parameter family of analytic varieties. A key tool in the proofs is a general subharmonicity property of certain functions associated to a uniform algebra. This property is given in Theorem 1.

1. Let X be a compact Hausdorff space, let A be a uniform algebra on X and let M be the maximal ideal space of A.

Fix $f \in A$. For each $\zeta \in \mathbf{C}$ put $f^{-1}(\zeta) = \{p \in M \mid f(p) = \zeta\}$ and for each subset Ω of $\mathbf{C}$, put $f^{-1}(\Omega) = \{p \in M \mid f(p) \in \Omega\}$. Consider an open subset Ω of $\mathbf{C}\setminus f(X)$. Supposing $f^{-1}(\Omega)$ to be nonempty, what can be said about the structure of $f^{-1}(\Omega)$? Work of Bishop **[2]** and Basener **[1]** yields that if $f^{-1}(\zeta)$ is at most countable for each $\zeta \in \Omega$, then $f^{-1}(\Omega)$ contains analytic disks. On the other hand, Cole **[4]** has given an example where no analytic disk is contained in $f^{-1}(\Omega)$. In §2 we prove.

THEOREM 1. *Let Ω be an open subset of $\mathbf{C}\setminus f(X)$. Choose $g \in A$. Define $Z(\zeta) = \sup_{f^{-1}(\zeta)} |g|$, $\zeta \in \Omega$. Then* $\log Z$ *is subharmonic in Ω.*

This theorem is proved by a method of Oka in **[5]**.

In §3 we apply Theorem 1 to the following situation: X is a compact set in $\mathbf{C}^2$, A is the uniform closure on X of polynomials in z and w. Here $M = h(X)$, the polynomially convex hull of X. We assume that X is invariant under the map T_θ:

$$(z, w) \rightarrow (e^{i\theta} z, e^{-i\theta} w) \quad \text{for} \quad 0 \leqq \theta < 2\pi.$$

Put $f = zw$. Let Ω be an open disk contained in $\mathbf{C} \setminus f(X)$ with $0 \notin \Omega$. Here $f^{-1}(\Omega) = \{(z, w) \in h(X) \mid zw \in \Omega\}$.

THEOREM 2. *If $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic disk.*

In §4, we consider the case when X is a disk in $\mathbf{C}^2$, defined:

$$X = \left\{(z, w) \,\middle|\, |z| \leqq 1,\ w = \frac{a(|z|)}{z}\right\},$$

where a is a continuous complex valued function defined on $0 \leqq r \leqq 1$, with $a(r) = o(r)$.

X is evidently invariant under T_θ for all θ. In Theorem 3 we give an explicit description of $h(x)$ for a certain class of such disks X.

2. *Proof of Theorem* 1. (Cf. **[5]**, §2.) Fix $\zeta_0 \in \Omega$ and let $\zeta_n \to \zeta_0$. Assume $Z(\zeta_n) \to t$. We claim $Z(\zeta_0) \geqq t$. For choose p_n in $f^{-1}(\zeta_n)$ with $|g(p_n)| = Z(\zeta_n)$. Let p be an accumulation point of $\{p_n\}$. Then $|g(p)| \geqq t$, whence $Z(\zeta_0) \geqq t$, as claimed. Thus Z is upper-semicontinuous at ζ_0, and so Z is upper-semicontinuous in Ω.

Theorem 1.6.3 of **[6]** gives that an upper-semicontinuous function u in Ω is subharmonic provided for each closed disk $D \subset \Omega$ and each polynomial P we have

$$u \leqq \operatorname{Re} P \quad \text{on } \partial D \quad \text{implies} \quad u \leqq \operatorname{Re} P \quad \text{on } D. \tag{1}$$

Fix a closed disk D contained in Ω and let $\mathring{D}$ be its interior. Choose a polynomial P such that $\log Z \leqq \operatorname{Re} P$ on ∂D. Then

$$Z(\zeta)\,|\exp(-P(\zeta))| \leqq 1 \quad \text{on } \partial D.$$

Hence for each ζ in ∂D, if x is in $f^{-1}(\zeta)$, then

$$|g(x)| \cdot |\exp(-P(f))(x)| \leqq 1, \quad \text{or} \tag{2}$$

$$|g \cdot \exp(-P(f))| \leqq 1 \quad \text{at} \quad x.$$

Now $g \cdot \exp(-P(f))$ is in A. Put $N = f^{-1}(\mathring{D})$. The boundary of N is contained in $f^{-1}(\partial D)$. Hence by the Local Maximum Modulus Principle for uniform algebras, for each y in N we can find x in $f^{-1}(\partial D)$ with

$$|g \exp(-P(f))(y)| \leqq |g \cdot \exp(-P(f))(x)|,$$

whence by (2) we have

$$|g \cdot \exp(-P(f))(y)| \leqq 1. \tag{3}$$

Fix ζ_0 in $\mathring{D}$. Choose y in $f^{-1}(\zeta_0)$ with $|g(y)| = Z(\zeta_0)$. Applying (3) to this y, we get

$$Z(\zeta_0)|\exp(-P(\zeta_0))| \leqq 1. \tag{4}$$

Hence $\log Z(\zeta_0) \leqq \operatorname{Re} P(\zeta_0)$. So (1) is satisfied, and so $\log Z$ is subharmonic in Ω, as desired.

3. *Proof of Theorem* 2. Since X is invariant under the maps T_θ, $h(X)$ is invariant under each T_θ. Fix $\zeta \in \Omega$. There are two possibilities:

(a) $|z|$ is constant on $f^{-1}(\zeta)$.

(b) $\exists$ r_1, r_2 with $0 < r_1 < r_2$ and $\exists$

$$(z_1, w_1), (z_2, w_2) \in f^{-1}(\zeta) \quad \text{with} \quad |z_1| = r_1, \ |z_2| = r_2.$$

Suppose (b) occurs. Then the circles: $z = r_1 e^{i\theta}$, $w = \zeta/r_1 e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$ and $z = r_2 e^{i\theta}$, $w = \zeta/r_2 e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$ both lie in $h(X)$. Hence the analytic annulus: $r_1 < |z| < r_2$, $w = \zeta/z$ lies in $f^{-1}(\zeta)$. Thus if (b) occurs at any point ζ in Ω, $f^{-1}(\Omega)$ does contain an analytic disk. Hence to prove the Theorem, we may assume that (a) holds for each $\zeta \in \Omega$. Define, for $\zeta \in \Omega$, $Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|$, $W(\zeta) = \sup_{f^{-1}(\zeta)} |w|$. Fix $(z_0, w_0) \in f^{-1}(\zeta)$. Since we have case (a), $Z(\zeta) = |z_0|$. Hence $W(\zeta) = |w_0|$ and so $Z(\zeta)W(\zeta) = |\zeta|$, whence

$$\log Z(\zeta) + \log W(\zeta) = \log |\zeta|.$$

Since $\log Z$ and $\log W$ are subharmonic in Ω while $\log |\zeta|$ is harmonic, $\log Z$, $\log W$ are in fact harmonic in Ω. Put $U = \log Z$ and let V be the harmonic conjugate of U in Ω. Put $\phi(\zeta) = e^{U+iV}(\zeta)$. Then ϕ is analytic in Ω and $|\phi| = Z$ in Ω.

Assertion. The variety $z = \phi(\zeta)$, $w = \zeta/\phi(\zeta)$, $\zeta \in \Omega$, is contained in $h(X)$.

Fix $\zeta \in \Omega$. Choose $(z_1, w_1) \in f^{-1}(\zeta)$. Then $Z(\zeta) = |z_1|$, so $|\phi(\zeta)| = |z_1|$, i.e., $\exists$ real α with $z_1 = \phi(\zeta)e^{i\alpha}$. Then $w_1 = \zeta/\phi(\zeta)e^{i\alpha}$. But $(e^{-i\alpha}z_1, e^{i\alpha}w_1) \in h(X)$. Hence $(\phi(\zeta), \zeta/\phi(\zeta) \in h(X)$. The Assertion is proved, and Theorem 2 follows.

Note. Questions related to the result just proved are studied by J. E. Björk in **[3]**.

4. Invariant disks in C^2. Let P be a polynomial with complex coefficients, $P(t) = \Sigma_{n=1}^N c_n t^n$, which is one-one on the unit interval with endpoints identified, i.e., we assume that $P(1) = P(0) = 0$ and $P(t_1) \neq P(t_2)$ if $0 \leqq t_1 < t_2 < 1$. Also assume $P'(t) \neq 0$ for $0 \leqq t \leqq 1$. Then the curve β given parametrically: $\zeta = P(t)$, $0 \leqq t \leqq 1$, is a simple closed analytic curve in the ζ-plane whose only singularity is a double-point at the origin. Denote by θ the angle between the two arcs of β meeting at 0. Assume $\theta < \pi$. Define $a(r) = P(r^2)$, i.e.,

$$a(r) = \sum_{n=1}^{N} c_n r^{2n}. \tag{5}$$

Let X be the disk in $\mathbf{C}^2$ defined

$$X = \left\{ \left(z, \frac{a(|z|)}{z} \right) \,\middle|\, |z| \leqq 1 \right\}. \tag{6}$$

The function $f = zw$ maps X on β. Denote by Ω the interior of β.

THEOREM 3. *$\exists$ function ϕ analytic in Ω such that $h(X)$ is the union of X and $\{(z,0) \,|\, |z| \leqq 1\}$ and*

$$\{(z,w) \,|\, zw \in \Omega \quad \textit{and} \quad |z| = |\phi(zw)|\}.$$

COROLLARY. *Every point of $h(X) \setminus X$ lies on some analytic disk contained in $h(X)$.*

NOTATION. $A(\Omega)$ denotes the class of functions F defined and continuous in $\bar{\Omega}$ and analytic in Ω.

$\mathfrak{A}$ denotes the algebra of functions on $|z| \leqq 1$ which are uniformly approximable by polynomials in z and $a(|z|)/z$.

LEMMA 1. *Let $G \in C[0, 1]$. If $G(|z|) \in \mathfrak{A}$, then $\exists F \in A(\Omega)$ such that $G(r) = F(a(r))$ for $0 \leqq r \leqq 1$.*

Proof. Let g be a polynomial in z and $a(|z|)/z$. Calculation gives that there is a polynomial $\tilde{g}$ in one variable with

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) d\theta = \tilde{g}(a(r)), \quad 0 \leqq r \leqq 1.$$

Choose a sequence $\{g_n\}$ of polynomials in z and $a(|z|)/z$ approaching

$G(|z|)$ uniformly on $|z| \leqq 1$. Then $\tilde{g}_n(a(r)) \to G(r)$ uniformly on $0 \leqq r \leqq 1$. Hence $\exists\, F \in A(\Omega)$ with $\tilde{g}_n \to F$ uniformly on β, so $G(r) = F(a(r))$.

LEMMA 2. *If $f = zw$, then $f^{-1}(\Omega)$ is not empty.*

Proof. Fix $\zeta_0 \in \Omega$. If $f^{-1}(\Omega)$ is empty, then $f - \zeta_0 \neq 0$ on $h(X)$ and so $(zw - \zeta_0)^{-1}$ lies in the closure of the polynomials in z and w on X. Then $(a(|z|) - \zeta_0)^{-1} \in \mathfrak{A}$. By Lemma 1, $\exists\, F \in A(\Omega)$ with $F(a(r)) = (a(r) - \zeta_0)^{-1}$. Then $(\zeta - \zeta_0)^{-1} \in A(\Omega)$, which is false. So $f^{-1}(\Omega)$ is not empty.

LEMMA 3. *Fix $\zeta \in \beta \setminus \{0\}$. Let (z_0, w_0) be a point in $h(X)$ with $z_0 w_0 = \zeta$. Then $(z_0, w_0) \in X$.*

Proof. Assume $(z_0, w_0) \notin X$. Let r be the point in $(0, 1)$ with $a(r) = \zeta$. Put, for each r, $\gamma_r = \{(re^{i\theta}, (a(r)/re^{i\theta})) \mid 0 \leqq \theta < 2\pi\}$. Then γ_r is a polynomially convex circle contained in X. Hence $\exists$ polynomial P with $|P(z_0, w_0)| > 2$, $|P| < 1$ on γ_r. Choose a neighborhood N of γ_r on X where $|P| < 1$. The image of $X \setminus N$ under the map $(z, w) \to zw$ is a closed subarc β_1 of β which excludes ζ. Choose $F \in A(\Omega)$ with $F(\zeta) = 1$, $|F| < 1$ on $\beta \setminus \{\zeta\}$. Then $\exists\, \delta > 0$ such that $|F| < 1 - \delta$ on β_1. Hence $|F(zw)| < 1 - \delta$ on $X \setminus N$. Also $|F(zw)| \leqq 1$ on X. Fix n and put

$$Q = F(zw)^n \cdot P(z, w).$$

$|Q(z_0, w_0)| > 2$. On N, $|Q| \leqq |P| < 1$. On $X \setminus N$, $|Q| < (1 - \delta)^n \cdot \max_X |P|$, and so $|Q| < 1$ on $X \setminus N$ for large n. Then $|Q| < 1$ on X. Since F is a uniform limit on β of polynomials in ζ, Q is a uniform limit on $X \cup \{(z_0, w_0)\}$ of polynomials in z and w. This contradicts that $(z_0, w_0) \in h(X)$. Thus $(z_0, w_0) \in X$. We are done.

Note. Since f maps X on β and $\mathbf{C} \setminus f(X)$ is the union of the interior and exterior of β, we conclude from the last Lemma that $h(X)$ is the union of X and $f^{-1}(\{0\})$ and $f^{-1}(\Omega)$.

We need some notation now. For each $\zeta \in \beta \setminus \{0\}$, denote by $r(\zeta)$ the unique r in $(0, 1)$ with $a(r) = \zeta$.

Since a is a polynomial in r vanishing at 0, there is a constant $d > 0$ such that

$$r(\zeta) > d\,|\zeta|, \quad \text{all} \quad \zeta \in \beta. \tag{7}$$

For $\zeta_0 \in \Omega$, denote by μ_{ζ_0} harmonic measure at ζ_0 relative to Ω. Since β consists of analytic arcs, with one jump-discontinuity for the tangent at $\zeta = 0$, $\mu_{\zeta_0} = K_{\zeta_0}\, ds$, where K_{ζ_0} is a bounded functions on β and ds is arc-length. Define

$$U(\zeta_0) = \int_\beta \log r(\zeta) d\mu_{\zeta_0}(\zeta).$$

Since (7) holds, this integral converges absolutely. U is a harmonic function in Ω, bounded above, and continuous at each boundary point $\zeta \in \beta \setminus \{0\}$ with boundary value $\log r(\zeta)$ at ζ.

For $\zeta \in \Omega$, define

$$Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|, \quad W(\zeta) = \sup_{f^{-1}(\zeta)} |w|.$$

LEMMA 4. *For all* $\zeta \in \Omega$, $\log Z(\zeta) \leqq U(\zeta)$ *and* $\log W(\zeta) \leqq \log|\zeta| - U(\zeta)$.

Proof. Fix $\zeta \in \beta \setminus \{0\}$, choose $\zeta_n \in \Omega$ with $\zeta_n \to \zeta$ and suppose $Z(\zeta_n) \to \lambda$. Choose $p_n \in f^{-1}(\zeta_n)$ with $Z(\zeta_n) = |z(p_n)|$. Without loss of generality, $p_n \to p$ for some point $p \in h(X)$. Then $f(p) = \zeta$. By Lemma 3, $p \in X$, i.e., $p = (\mathrm{re}^{i\theta}, (a(r)/\mathrm{re}^{i\theta}))$ for some r, θ. Also $a(r) = \zeta$ and so $r = r(\zeta)$, whence $|z(p_n)| \to r(\zeta)$ and so $\lambda = r(\zeta)$. Thus $Z(\zeta') \to r(\zeta)$ as $\zeta' \to \zeta$ from within Ω, and so $\log Z$ assumes the same boundary values as U, continuously on $\beta \setminus \{0\}$.

For each positive integer k, let $\Omega_k = \{\zeta \in \Omega \,|\, |\zeta| > 1/k\}$. $\partial\Omega_k$ is the union of a closed subarc β_k of $\beta \setminus \{0\}$ and an arc α_k on the circle $|\zeta| = 1/k$.

Fix $\zeta_0 \in \Omega$. For large $k, \zeta_0 \in \Omega_k$. Denote by $\mu_{\zeta_0}^{(k)}$ the harmonic measure at ζ_0 relative to Ω_k. An elementary estimate gives that there is a constant C_{ζ_0} independent of k such that

$$\mu_{\zeta_0}^{(k)}(\alpha_k) \leqq C_{\zeta_0} \cdot \frac{1}{\sqrt{k}} \text{ for all } k. \tag{8}$$

Let S be any function subharmonic in Ω and assuming continuous boundary values, again denoted S, on $\beta \setminus \{0\}$. Assume $\exists$ constant M with $S \leqq M$ in Ω. Then for all k,

$$S(\zeta_0) \leqq \int_{\beta_k} S d\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} M d\mu_{\zeta_0}^{(k)}, \quad \text{whence} \tag{9}$$

$$S(\zeta_0) \leqq \int_{\beta_k} S d\mu_{\zeta_0}^{(k)} + M \cdot C_{\zeta_0} \cdot \frac{1}{\sqrt{k}}.$$

Applying (9) with $S = \log Z$, we get

$$\log Z(\zeta_0) \leqq \int_{\beta_k} U d\mu_{\zeta_0}^{(k)} + MC_{\zeta_0} \cdot \frac{1}{\sqrt{k}}, \tag{10}$$

since as we saw earlier, $\log Z = U$ on $\beta \setminus \{0\}$.

By (7), if $\zeta' \in \alpha_k$,

$$U(\zeta') = \int_\beta \log r(\zeta) d\mu_{\zeta'}(\zeta) > C + \int_\beta \log |\zeta| \, d\mu_{\zeta'}(\zeta),$$

where C is a constant, so

$U(\zeta') > C + \log |\zeta'| = C + \log \dfrac{1}{k}$. Hence

$$\begin{aligned} U(\zeta_0) &= \int_{\beta_k} U d\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} U d\mu_{\zeta_0}^{(k)} \\ &\geqq \int_{\beta_k} U d\mu_{\zeta_0}^{(k)} + \left(C + \log \frac{1}{k}\right) \frac{C_{\zeta_0}}{\sqrt{k}}. \end{aligned}$$

Combining this with (10) and letting $k \to \infty$, we get that $\log Z(\zeta_0) \leqq U(\zeta_0)$, as desired. A parallel argument gives the assertion regarding W. We are done.

LEMMA 5. *With Z defined as above,* $\log Z(\zeta) = U(\zeta)$ *for all* $\zeta \in \Omega$, *and* $\log W(\zeta) = \log |\zeta| - U(\zeta)$.

Proof. Suppose either equality fails at some point ζ_0. By the last Lemma, this implies that

$$\log Z(\zeta_0) + \log W(\zeta_0) < \log |\zeta_0|.$$

Fix $p \in f^{-1}(\zeta_0)$. Then $|z(p)| \leqq Z(\zeta_0)$, $|w(p)| \leqq W(\zeta_0)$, so

$$\log |z(p) w(p)| < \log |\zeta_0|.$$

But $z(p)w(p) = \zeta_0$, so we have a contradiction, proving the Lemma.

Proof of Theorem 3. Let V denote the harmonic conjugate of U in Ω and put $\phi = e^{U+iV}$. Fix $(z_0, w_0) \in f^{-1}(\Omega)$ and put $\zeta_0 = z_0 \cdot w_0$. Unless $|z_0| = Z(\zeta_0)$ and $|w_0| = W(\zeta_0)$, we have

$$|\zeta_0| = |z_0| \, |w_0| < Z(\zeta_0) W(\zeta_0) = |\zeta_0|$$

by the last Lemma. So we must have $|z_0| = Z(\zeta_0) = |\phi(\zeta_0)|$.

Conversely fix $\zeta_0 \in \Omega$ and let (z_0, w_0) be a point in $\mathbf{C}^2$ such that $z_0 \cdot w_0 = \zeta_0$ and $|z_0| = |\phi(\zeta_0)|$. Choose $(z_1, w_1) \in f^{-1}(\zeta_0)$. By the preceding $|z_1| = |\phi(\zeta_0)|$, so $\exists$ real α with $z_0 = e^{i\alpha} z_1$, $w_0 = e^{-i\alpha} w_1$. Hence $(z_0, w_0) \in h(X)$, so $(z_0, w_0) \in f^{-1}(\Omega)$. Thus $f^{-1}(\Omega)$ consists precisely of those points (z, w) with $zw \in \Omega$ and $|z| = |\phi(zw)|$.

To finish the proof we need only identify $f^{-1}(0)$. The circle $\{(z, 0) \mid |z| = 1\}$ lies in X, so the disk D: $\{(z, 0) \mid |z| \leqq 1\}$ is contained in $f^{-1}(0)$. If $(z_0, w_0) \in f^{-1}(0)$ and does not lie in D, then $z_0 = 0$, $w_0 \neq 0$. The same argument as was used in proving Lemma 3 shows that then $(z_0, w_0) \notin h(X)$, contrary to assumption. So $f^{-1}(0) = D$, and the proof of Theorem 3 is finished.

Remark. As we have just seen, $f^{-1}(\Omega)$ is the union of varieties V_α, $0 \leqq \alpha < 2\pi$, where V_α is defined:

$$z = e^{i\alpha} \phi(\zeta), \quad w = e^{-i\alpha} \frac{\zeta}{\phi(\zeta)}, \quad \zeta \in \Omega.$$

What does the boundary of such a variety V_α in $h(X)$ look like? It splits into two sets:

$$S = \{(z, w) \in \partial V_\alpha \mid zw \in \beta \setminus \{0\}\} \quad \text{and}$$

$$T = \{(z, w) \in \partial V_\alpha \mid zw = 0\}.$$

It is easy to see that S is an arc on X cutting each circle: $\{(z, w) \in X \mid |z| = r\}$, $0 < r < 1$, exactly once while T is a closed subset of the disk $D = \{(z, 0) \mid |z| \leqq 1\}$.

It is remarkable that even though X is itself very regular, the rest of the hull of X is attached to X in a very complicated way.

Acknowledgements. I am grateful to Andrew Browder and Bruce Palka for their helpful suggestions.

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Received April 23, 1974.

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