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RINGS WITH QUASI-PROJECTIVE LEFT IDEALS

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A ring R is a left qp-ring if each of its left ideals is quasi-projective as a left R-module in the sense of Wu and Jans. The following results giving the structure of left qp-rings are obtained. Throughout R is a perfect ring with radical N: (1) Let R be local. Then R is a left qp-ring iff $N^2 = (0)$ or R is a principal left ideal ring with dcc on left ideals, (2) If R is a left qp-ring and T is the sum of all those indecomposable left ideals of R which are not projective, then T is an ideal of R and $N = T \oplus L$, L is a left ideal of R such that every left subideal of L is projective, R/T is hereditary, and R is heredity iff $T = (0)$. (3) If R is left qp-ring then $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$, where S is hereditary, T is a direct sum of finitely many local qp-rings and M is a (S, T)-bimodule. (4) A perfect left qp-ring is semi-primary. (5) Let R be an indecomposable ring such that it admits a faithful projective injective left module. Then R is a left qp-ring iff R is a local principal left ideal ring or R is a left-hereditary ring with dcc on left ideals. (6) Let R be an indecomposable QF-ring. Then R is a left qp-ring if each homomorphic image of R is a q-ring (each one-sided ideal is quasi-injective). (7) If a left ideal A of left qp-ring R is not projective then the projective dimension of A is infinite, thus $lgl.$ $\dim R = 0, 1$, or ∞. An example of a left artinian left qp-ring which is not right qp-ring is also given.

Clearly all left hereditary rings are left qp-rings. However, the class of commutative principal ideal artinian rings which are not direct sum of fields distinguishes qp-rings from hereditary rings. Commutative pre-self-injective rings studied by Klatt and Levy **[8]** and by Levy **[11]** form a class dual to the class of commutative qp-rings. Dual to the noncommutative qp-rings are rings for which every cyclic module is quasi-injective investigated by Ahsan **[1]** and by Koehler **[9]**. In this paper we study perfect left qp-rings.

2. A ring R is said to be right (left) perfect if it satisfies dcc on principal left (right) ideals and R is called perfect if it is both right and left perfect **[3]**. An artinian principal ideal ring is called uniserial.

A ring R with Jacobson radical N is called local if R/N is a division ring. We assume that all nonzero rings have nonzero identity elements and all modules are unital. An R-module M is said to be quasi-projective if for every submodule N of M, the induced sequence $0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, M) \to \mathrm{Hom}(M, M/N) \to 0$ is exact. For basic properties of quasi-projective modules we refer to Wu and Jans **[14]**. Quasi-injective modules are defined dually in **[7]**. The following theorems give the structure of a quasi-projective module over a perfect ring.

THEOREM 1. (Wu and Jans **[14]**). *A finitely generated indecomposable quasi-projective left module over a right perfect ring R is of the form Re/Ae where e is a primitive idempotent and A is an ideal of R* (Indeed the theorem is proved when R is semi-perfect.).

THEOREM 2. (Koehler **[10]**). *Let R be a right perfect ring. A left R-module M is quasi-projective if and only if*

$$M = \bigoplus \sum_{i=1}^{k} (Re_i/Ae_i)^{g(i)}$$

Where A is an ideal and $e_1, e_2, \cdots, e_k$ are indecomposable orthogonal idempotents; the number of nonisomorphic simple R-modules is k, and $Re_1, Re_2, \cdots, Re_k$ are the corresponding nonisomorphic projective covers. In addition the decomposition is unique upto automorphism.

As defined by Miyashita **[12]**, a module M is called perfect if for any pair of submodules A, B of M with $A + B = M$ there exists a submodule B_0 of B that is minimal with respect to the property that $A + B_0 = M$. In this case B_0 is called a d-complement of A (in M).

THEOREM 3. (Miyashita **[12]**). *If every homomorphic image of a module M has a projective cover then M is perfect. Further if M is perfect and quasi-projective then the sum of two submodules of M which are d-complements of each other is direct.*

Finally, in this section we state a lemma which is analogous to the lemma in Rangaswamy and Vaneja **[13]**.

LEMMA 1. *Let $A \oplus B$ be a quasi-projective left R-module. Then every epimorphism from A to B splits.*

3. In all the Lemmas 2, 3, 4, 5 and 6 which follow it is assumed that R is a perfect left qp-ring and we write $R = Re_1 \oplus \cdots \oplus Re_n$, where $\{e_i\}_{1 \leq i \leq n}$ are primitive orthogonal idempotents. Denote the Jacobson radical of R by N.

LEMMA 2. *Let A and B be two indecomposable left ideals of R. Then either $A \cap B = (0)$ or A and B are comparable.*

Proof. Let $A \not\subset B$ and $B \not\subset A$. As $A + B$ is quasi-projective perfect left R-module, by Theorem 3, there exist nonzero left subideals A_0 of A and B_0 of B such that $A + B = A_0 \oplus B_0$. Then $A = A_0 \oplus (A \cap B_0)$ yields that $A = A_0$ as A is indecomposable. Similarly $B = B_0$. Hence $A \cap B = (0)$.

LEMMA 3. *If an indecomposable left ideal A is not projective then for some i, $A \subset Ne_i$ and $A = Re_ixe_i$ for some $e_ixe_i \in e_iNe_i$; further for this i, $Re_iRe_j = (0)$ for all $j \neq i$. In particular, if $eNe \neq (0)$ then* $\hom(Re, Rf) = 0$ *for all primitive idempotents f not equal to e. Also conversely, any left ideal of the form $A = Re_ixe_i$, $e_ixe_i \in e_iNe_i$ is an indecomposable nonprojective left ideal.*

Proof. By Theorem 1, $A \cong Re_i/Ie_i$ for some ideal I of R. If $A \not\subset Re_i$, then by Lemma 2, $A \cap Re_i = (0)$. But then the left ideal $Re_i \oplus A$ of R is quasi-projective and there exists an epimorphism $\sigma: Re_i \to A$ which must split by Lemma 1. So σ is an isomorphism and A is projective which is a contradiction. Hence $A \subset Ne_i$, since Ne_i is the unique maximal left subideal of Re_i. Further, A being a homomorphic image of Re_i, $A = Re_ixe_i$ for some $e_ixe_i \in e_iNe_i$. For proving $Re_iRe_j = 0$, $i \neq j$, let us assume that for some j, $Re_iRe_j \neq 0$. So there exists $a \in R$ such that $Re_iae_j \neq 0$. As $Re_i \oplus Re_iae_j$ is quasi-projective, Lemma 1 yields that $Re_i \cong Re_iae_j$. Then $Re_iae_j \oplus A$ is quasi-projective and A is a homomorphic image of Re_iae_j. Consequently, Lemma 1 gives that A is projective which is a contradiction. Hence for all $j \neq i$, $Re_iRe_j = (0)$.

LEMMA 4. *For a fixed i, either the family of all nonzero left ideals of the form Re_iae_i, $e_iae_i \in e_iNe_i$ are isomorphic or $Re_iNe_i = Re_ine_i$ for some $e_ine_i \in e_iNe_i$.*

Proof. Since R is a perfect ring, N is both right and left T-nilpotent. We assert that there exists a maximal left ideal in the family $F = \{Re_ixe_i \mid e_ixe_i \in e_iNe_i\}$. For if Re_ibe_i is not maximal then we can find $Re_ib_1e_i \supset Re_ibe_i$. This gives $e_ibe_i = (e_ix_1e_i)(e_ib_1e_i)$ with $e_ix_1e_i \in e_iNe_i$. If $Re_ib_1e_i$ is not maximal then we can find $Re_ib_2e_i \supset Re_ib_1e_i \supset Re_ibe_i$. This yields $e_ibe_i = (e_ix_2e_i)(e_ib_1e_i)$ and thus $e_ibe_i = (e_ix_2e_i)(e_ix_1e_i)(e_ib_ie_i)$. By continuing this process, we get a sequence $(e_ix_je_i)$, $j = 1, 2, \cdots$, with $e_ix_je_i \in e_iNe_i$. Since N is T-nilpotent this sequence cannot be infinite. Hence we can find a maximal left ideal, say, Re_ine_i in the family F. We claim that either $Re_iNe_i = Re_ine_i$ or all left ideals of the form Re_iae_i, $e_iae_i \in e_iNe_i$ are isomorphic. So if $Re_iNe_i \neq Re_ine_i$ then there exists some $x \in N$ such that $Re_ixe_i \not\subset Re_ine_i$. Then by Lemma 2, $Re_ixe_i \cap Re_ine_i = (0)$. Let $A = Re_ine_i \oplus Re_ixe_i$. A is a quasi-projective left ideal of R and both Re_ine_i, Re_ixe_i have same projective cover Re_i. So by Theorem 2, $Re_ine_i \cong Re_ixe_i$. We now show for every $a \in N$, Re_iae_i is isomorphic to Re_ine_i. By Lemma 2 and maximality of Re_ine_i, Re_iae_i must have zero intersection with one of the two left ideals Re_ine_i, Re_ixe_i. In either case we get by invoking Theorem 2 again that $Re_iae_i \cong Re_ine_i$. Hence all left ideals of the form Re_iae_i are isomorphic as desired. This completes the proof.

LEMMA 5. *For a fixed i either* $(e_iNe_i)^2 = (0)$ *or* e_iRe_i *is a principal left ideal ring with dcc* (all proper left ideals are powers of e_iNe_i) *and all left subideals of* Re_i *generated by subsets of* e_iNe_i *satisfy dcc.*

Proof. There is a 1–1 inclusion preserving correspondence between all left ideals of e_iRe_i and all those left subideals of Re_i which are generated by subsets of e_iNe_i. If, as in the Lemma 4, all nonzero principal left subideals of Re_i of the form Re_iae_i, $e_iae_i \in e_iNe_i$ are isomorphic, we derive that all the principal left subideals of e_iNe_i in e_iRe_i are isomorphic and hence minimal. Consequently, $(e_iNe_i)^2 = (0)$. In the other case we have $Re_iNe_i = Re_ine_i$. This implies $e_iRe_iNe_i = e_iRe_ine_i$ and so $e_iNe_i = e_iRe_ine_i$. Thus in the local ring e_iRe_i, the radical is a principal left ideal generated by a nilpotent element. This yields that all the left ideals of e_iRe_i are of the form $e_iRe_i(e_ine_i)^t (= (e_iNe_i)^t)$, $t = 1, 2, \cdots, k$, where k is the index of nilpotency of e_iNe_i. But then this gives that all the left subideals of Re_i generated by the subsets of e_iNe_i are of the form $R(e_ine_i)^t$. This completes the proof.

THEOREM 4. *Let R be a perfect left qp-ring. Then for any primitive idempotent e of R, eRe is also a left qp-ring.*

Proof. Let $R = Re_1 \oplus \cdots \oplus Re_n$, where e_i are primitive orthogonal idempotents. Without loss of generality we can suppose that $e = e_1$. Let $N = J(R)$ be the Jacobson radical. If $(e_1Ne_1)^2 = (0)$, then e_1Ne_1 is a completely reducible left e_1Re_1-module. Trivially then every left ideal of e_1Re_1 is quasi-projective. Suppose $(e_1Ne_1)^2 \neq 0$. By Lemma 5, any proper left-ideal of e_1Re_1 is a power of e_1Ne_1, and thus it is isomorphic to $e_1Re_1/(e_1Ne_1)^t$ for some positive integer t which is quasi-projective. Hence e_1Re_1 is a left qp-ring.

Combining Theorem 4 and the above lemmas we obtain:

THEOREM 5. *Let R be a local perfect ring. Then R is a left qp-ring if and only if*

(i) $N^2 = (0)$, *or*

(ii) *R is a principal left ideal ring with dcc on left ideals.*

Next we prove a proposition which is also of an independent interest.

PROPOSITION 1. *Let R be a left perfect ring. If every left ideal contained in the radical N is projective, then R is left hereditary.*

Proof. Since idempotents modulo the radical can be lifted, given any left ideal I of R, $I = Rf_1 \oplus \cdots \oplus Rf_n \oplus J$, for some idempotents $f_1, \cdots, f_n$ and for some left ideal $J \subset N$. By hypothesis J is projective. Hence I is projective and so R is left hereditary.

LEMMA 6. *Any nonzero left subideal of Ne (e* primitive idempotent) *of the form $Reae$ in a perfect ring R cannot have nonzero homorphism into any indecomposable left ideal B which is a homomorphic image of some Rf with f, a primitive idempotent, such that $Rf \neq Re$.*

Proof. Let $A \doteq Reae$. Since $eNe \neq (0)$, by Lemma 3, $ReRf = (0)$, where f is a primitive idempotent not equal to e. Since A is not projective, each of its nonzero homomorphic image is also not projective. So let B be an indecomposable homomorphic image of $A = Reae$. Since B is an indecomposable quasi-projective (but not projective) left ideal, by theorem 1, B is of the form Rf/Xf where $Xf \neq 0$ and f is some primitive idempotent. We wish to show that $f = e$. By Lemma 3, $B \subset Rf$. But then we get a nonzero homomorphism $Re \to Reae \to B \to Rf$ which is a contradiction unless $e = f$. Thus $Reae$ cannot map onto any Rf/Xf with $Rf \neq Re$. This completes the proof.

THEOREM 6. *Let R be a perfect left qp-ring and let e_i, $1 \leqq i \leqq n$, be a maximal set of primitive orthogonal idempotents in R. Suppose $T = \Sigma_{i=1}^{n} Re_iNe_i$. Then* (i) *$T$ is the sum of all those indecomposable left ideals of R which are not projective,* (ii) *T is an ideal of R contained in N, and* (iii) *$N = T \oplus L$ for some left ideal L of R such that every left subideal of L is projective.*

Proof. By Lemma 6, $Re_iNe_iRe_j = (0)$ for $i \neq j$. So T is an ideal of R. Also, by Lemma 3, an indecomposable left ideal A of R is not projective if and only if $A = Re_iae_i$ for some $0 \neq e_iae_i \in e_iNe_i$. Thus it is immediate that T is the sum of all nonprojective indecomposable left ideals of R. We now proceed to prove (iii). Since Ne_i is quasi-projective, we can write $Ne_i = \oplus\Sigma B_k$, where B_k are indecomposable left subideals of Ne_i (Theorem 2). Consider $0 \neq e_ixe_i \in e_iNe_i$. Then Re_ixe_i has nonzero projection into some B_k. By Lemma 6, B_k itself is of the type Re_iye_i, $e_iye_i \in e_iNe_i$. It follows from Lemma 3 that Re_iNe_i is a sum of those indecomposable left ideals B_k which are homomorphic immages of Re_i. Also if some B_k is not homomorphic image of Re_i, then this B_k must be projective. Hence we can write $Ne_i = Re_iNe_i \oplus C_i$ where C_i is projective. This gives $N = T \oplus C$ where C is projective. Consider a left ideal $B(\neq 0)$ contained in C. Now $B = \oplus\Sigma X_\alpha$, where X_α are indecomposable left ideals. If some X_α is not projective, then by Lemma 3, X_α is of type Re_ixe_i with e_ixe_i in e_iNe_i and hence $X_\alpha \subset T$ which is a contradiction. This shows that B is projective, thus proving the theorem.

THEOREM 7. *Let R be a perfect left qp-ring, and T be the ideal as in Theorem 6. Then R/T is left hereditary and R is left hereditary if $T = (0)$.*

Proof. Consider a left ideal $A/T \subseteq N/T$. Since $N = T \oplus C$, we get $A = T \oplus (A \cap C)$. But all left subideals of C are projective. So $A \cap C$ is projective as a left R-module. Also $T(A \cap C) = (0)$ gives that $A \cap C$ is projective as left R/T-module. Then by Proposition 1, R/T is left hereditary. The last assertion in the theorem is obvious. This completes the proof.

The next theorem gives us a representation of a perfect left qp-ring as a triangular matrix ring.

THEOREM 8. *Let R be a left, right perfect left qp-ring. Then*

(1) *R is semi-primary*
(2) *R is an upper-triangular matrix ring of the form*

$$\begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$$

where S is a hereditary semi-primary ring, T is a finite direct sum of local left qp-rings, and M is an (S, T)-bimodule such that ${}_SM$ is projective.

Before we prove this theorem we establish some preliminaries and prove three lemmas. Let R be a perfect left qp-ring and N be its radical. Let $Rf_1, Rf_2, \cdots, Rf_m$ be a maximal set of nonisomorphic indecomposable left ideals of R generated by primitive idempotents. By invoking Lemma 1 we note that any nonzero R-

homomorphism of Rf_i into Rf_j is a monomorphism. Define a relation $\leqq$ in the set $\{Rf_1, \cdots, Rf_m\}$ as follows: $Rf_i \leqq Rf_j$ if and only if there exists a nonzero R-homomorphism of Rf_i into Rf_j, that is, $f_iRf_j \neq (0)$. By using the fact that in a right perfect ring R no principal left ideal Ra of R can be isomorphic to its own proper left subideal, we get that $\{Rf_1, \cdots, Rf_m\}$ is a partially ordered set with respect to $\leqq$. Further, recall Lemma 3 which says that for a given primitive idempotent f, either all left subideals of Rf are projective or $fNf \neq (0)$ and for any primitive idempotent e with $Rf \neq Re, fRe = (0)$. So if for some f_i, Rf_i has a left subideal which is not projective then $Rf_j \not\leqq Rf_i$ for all $i \neq j$. Hence we can arrange $Rf_1, \cdots, Rf_m$ in such a way that there exists a positive integer u (possibly zero) which is less than or equal to m satisfying the following:

(i) $f_jRf_i = (0)$ for $i < j$.

(ii) Every left subideal of Rf_i is projective and f_iRf_i is a division ring for $i \leqq u$.

(iii) $f_jNf_j \neq (0)$ and $f_jRf_i = (0)$ for $j > u$ and $i \neq j$.

Write

$$R = (Rf_{11} \oplus \cdots \oplus Rf_{1t_1}) \oplus (R_{21} \oplus \cdots \oplus Rf_{2t_2}) \oplus \cdots \oplus (Rf_{m_1} \oplus \cdots \oplus Rf_{mt_m})$$

where f_{ij} are orthogonal primitive idempotents with their sum equal to 1 such that $Rf_{ik} \cong Rf_i$ for every k and i. Clearly, by what is stated above, $t_i = 1$ for $i \geqq u + 1$; and $f_{ik}Rf_{ik}$ is a division ring whenever $i \leqq u$. Let $E_i = \Sigma_{k=1}^{t_i} f_{ik}$, $1 \leqq i \leqq m$ and $E = \Sigma_{i=1}^{u} E_i$. Then we have the following:

LEMMA 7. (1) *For $i \leqq u, E_iRE_i$ is simple artinian.*

(2) *$E_jRE_i = (0)$ whenever $i < j$.*

(3) *N is nilpotent.*

Proof. Since $Rf_{ik} \cong Rf_i, 1 \leqq k \leqq t_i$ and $RE = \oplus\Sigma_{k=1}^{u} Rf_{ik}$, we get E_iRE_i is anti-isomorphic to the $t_i \times t_i$ matrix ring $D_{t_i}^{(i)}$ where $D^{(i)} = f_iRf_i$ is a division ring. This proves (1).

The proof of (2) is immediate consequence of the fact that $f_jRf_i = (0)$ for $i < j$.

Finally, to prove (3), let $A = \Sigma_{i<j} E_iRE_j$. Then A is a nilpotent ideal and

$$R/A \cong \oplus\sum_{i=1}^{u} E_iRE_i \oplus \sum_{i=u+1}^{m} E_iRE_i$$
$$= \oplus\sum_{i=1}^{u} E_iRE_i \oplus \sum_{i=u+1}^{m} f_{i1}Rf_{i1}.$$

Since each $E_iRE_i, 1 \leqq i \leqq u$, is simple artinian and by Theorems 4 and 5 each $f_{i1}Rf_{i1}, u + 1 \leqq i \leqq m$ is a local ring with nilpotent maximal ideal, we obtain that the radical of R/A is nilpotent. Hence N is nilpotent since A is nilpotent.

LEMMA 8. *$S = ERE$ is hereditary.*

Proof. Since $V = \Sigma_{i<j\leqq u} E_iRE_j$ is the radical of S and S is semiprimary, in order to prove S is hereditary it is enough to prove that ${}_SV$ is projective. Now

$$V = \bigoplus \sum_{i<j\leqq u} E_iNE_j = \bigoplus \sum_{j=1}^{u} ENE_j = \bigoplus \sum_k \sum_{j=1}^{u} ENf_{jk}.$$

Also by our arrangement Nf_{jk} is projective as left R-module whenever $j \leqq u$. Thus ENf_{jk} is projective as left ERE-module and hence ${}_SV$ is projective as desired.

LEMMA 9. *$M = ER(1-E)$ is a projective left ERE-module.*

Proof. Consider

$$A = RER(1-E) = \sum_\alpha \sum_{i\leqq u} \sum_k Rf_{ik}a,$$

$a \in R(1-E)$. Hence A is a homomorphic image of a projective module $P = \bigoplus \Sigma_{a\in ER(1-E)} \Sigma_k \Sigma_{i\leqq u} X_{ika}$ where $X_{ika} \cong Rf_{ik}$ for $a \in R(1-E)$. Now A has a projective cover $Q = \bigoplus \Sigma_{\alpha\in\Lambda} X_\alpha$ such that each $X_\alpha \cong Rf_{i(\alpha)}$, $1 \leqq i(\alpha) \leqq m$. As A is a left ideal of R, A is quasi-projective. So by Koehler's theorem (Theorem 2) there exists an ideal $B \subset N$ such that $A = \bigoplus \Sigma_{\alpha\in\Lambda} Y_\alpha$, $Y_\alpha \cong (R/B)\bar{f}_{i(\alpha)}$. Since Q is a projective cover of A, Q is a direct summand of P. Thus, for each $i(\alpha)$, there exists a nonzero R-homomorphism of $Rf_{i(\alpha)}$ into one of Rf_i with $i \leqq u$. This along with (2) of Lemma 7 yields that $i(\alpha) \leqq u$ for all α. Since $Y_\alpha \subseteq R(1-E)$ and $i(\alpha) \leqq u$, an application of the Lemma 1 gives that the canonical homomorphism $Rf_{i(\alpha)} \to (R/A)\bar{f}_{i(\alpha)} = Y_\alpha$ is an isomorphism. Hence $Y_\alpha \cong X_\alpha$ for all α and A is projective.

Proof of the Theorem 8. Since N is nilpotent, R is semiprimary. Further, write

$$R = ERE \oplus ER(1-E) \oplus (1-E)R(1-E).$$

By the above lemmas $S = ERE$ is hereditary and $M = ER(1-E)$ is a projective left S-module. Also $T = (1-E)R(1-E) = \bigoplus \Sigma_{i=u+1}^{m} f_iRf_i$ is a direct sum of local left qp-rings. Hence $R \cong \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ where S, T and M are as stated in the theorem.

4. In this section we prove a theorem for a perfect left qp-ring which admits a uniform projective left module. This theorem then enables us to characterize perfect left qp-rings which admit a faithful projective injective left module (Theorem 10). We begin with

THEOREM 9. *Let M be a uniform projective left module over a perfect left qp-ring R. Then $M \cong Re_i$ for some primitive idempotent e_i, and either* (i) *All left subideals of Ne_i are homomorphic image of Re_i and $Re_iRe_j = (0) = Re_jRe_i$ where e_j is a primitive idempotent such that $Re_i \neq Re_j$, or* (ii) *Ne_i is projective and all its left ideals are projective. In each case Re_i satisfies dcc on left subideals.*

Proof. $M \cong Re_i$ follows from well known result of Bass **[3]**. As in the proof of Theorem 6, we can write $Ne_i = Re_iNe_i \oplus B_1$ where B_1 is projective. Since Re_i is uniform either $Ne_i = Re_iNe_i$ or $Ne_i = B_1$. In case $Ne_i = Re_iNe_i$, there exists $e_ine_i \in e_iNe_i$ such that $Ne_i = Re_ine_i$. Since e_ine_i is nilpotent, we get that every left subideal of Re_i is of the form $R(e_ine_i)^t = (Re_ine_i)^t$ which is obviously a homomorphic image of Re_i. It also follows that Re_i has only a finite number of left subideals. By Lemma 3, we know $Re_iRe_j = (0)$ where $Re_i \neq Re_j$. We show that Re_jRe_i is also zero. Suppose not then we can choose $xe_i \in Re_i$ with $Re_jxe_i \neq 0$ and $Re_jxe_i \subset Re_i$. Lemma 1 yields that Re_jxe_i is projective and thus $Re_jxe_i \cong Re_i$. But this is a contradiction since R is perfect. This proves (i).
(In the other case we have $Ne_i = B_1$ and B_1 is projective (also uniform). So B_1 is isomorphic to some Re_j. Also by theorem 6 every subideal of B_1 is projective and hence isomorphic to some Re_k. Further by Lemma 2 it follows that the subideals of Ne_i are totally ordered. Since no subideal ($\neq Re_j$) of Re_j can be isomorphic to Re_j, we conclude that there are only a finite number of subideals of Ne_i. This completes the proof.

The next theorem characterizes perfect left qp-rings admitting a faithful projective injective module.

THEOREM 10. *Let R be an indecomposable* (as a ring) *perfect ring such that it admits a faithful projective injective left R-module M. Then R is a left qp-ring if and only if*

(i) R is a local principal left ideal ring, or
(ii) *R is a left hereditary ring with dcc on left ideals*

Proof. Sufficiency is obvious. So let R be a left qp-ring. If we write $R = Re_1 \oplus \cdots \oplus Re_n$, e_i primitive orthogonal idempotents, then by Bass [3] M is a direct sum of copies of Re_i's, say, $Re_{t+1}, \cdots, Re_n$. Then

$A = Re_{t+1} \oplus \cdots \oplus Re_n$ is a faithful injective projective left R-module. We claim that each Re_i, $1 \leqq i \leqq n$, is uniform. If $i \geqq t+1$ then it is clear that Re_i is uniform. So let $i < t$. As A is faithful, $e_iRe_j \neq (0)$ for some $j \geqq t+1$. By using Lemma 1, we get that Re_i is isomorphic to a left subideal of Re_j. Hence Re_i is uniform, since Re_j is uniform.

Now by Theorem 9, Re_i satisfies *dcc*. It is also clear from the proof of that theorem that each of the left subideals in Re_i is principal. Hence R satisfies *dcc* on left ideals. In case $n = 1$, R is of type (i). So consider the case when $n > 1$. We claim Ne_i is projective. For if Ne_i is not projective, then by Theorem 9, Re_i and $\Sigma_{j \neq i} Re_j$ are two nonzero ideals and $R = Re_i \oplus \Sigma_{j \neq i} Re_j$. This contradicts the assumption that R is indecomposable. Hence Ne_i is projective. So $N = \oplus \Sigma Ne_i$ is projective as a left R-module. Hence R is left hereditary left artinian. This completes the proof.

As a special case of the above theorem we have the following characterization of QF-rings.

THEOREM 11. *Let R be an indecomposable QF-ring. Then R is a left qp-ring iff each homomorphic image of R is a q-ring* (each one-sided ideal is quasi-injective).

Proof. Since a left hereditary QF-ring is semisimple artinian, Theorem 10 gives that either R is simple or local uniserial. In a local uniserial ring every one sided ideal is two sided and every homomorphic image is QF-ring. Consequently, every homomorphic image is a q-ring **[6]**.

Conversely, if every homomorphic image of R is a Q-ring then also R is uniserial (R is uniserial iff every homomorphic image of R is QF, Fuller **[5]**). Further R is isomorphic to a full $n \times n$ matrix ring over a local ring B. If $n = 1$ then R is local uniserial. If $n > 1$ then R must be simple artinian, since R is a q-ring (c.f. Jain, Mohamed and Singh **[6]**, Theorem 2.4) **[6]**. In each case R is a left qp-ring. This completes the proof.

5. In this section we study left global dimension of a perfect left qp-ring.

THEOREM 12. *Let R be a perfect left qp-ring and A be a left ideal of R. Then the projective dimension of A as a left R-module is* 0 *or* ∞.

Proof. We first prove a sublemma.

SUBLEMMA. *Under the hypothesis of the theorem if e is a primitive idempotent and $0 \neq exe \in eNe$ and 1_R (exe) denotes the left annihilator of exe in R then 1_R (exe) $= L \oplus M$, where $L = Reye, 0 \neq eye \in eNe$, is not projective.*

Proof of the sublemma. By Theorem 2 we can write 1_R (exe) $= \oplus \Sigma A_\alpha$ where A_α are indecomposable left ideals. Also it follows from Lemma 5 that 1_R (exe) $\cap eNe \neq 0$. Let us choose $0 \neq eue \in 1_R$ (exe) $\cap$ eNe. Then $Reue$ has nonzero homomorphism into one of A_α's. By Lemma 6, $A_\alpha = Reye$ for some eye in eRe. Indeed $eye \in eNe$ since $Re \not\subset 1_R$ (exe). Hence A_α is not projective. This completes the proof of the sublemma.

We now prove the theorem. Since A is a direct sum of indecomposable left ideals (Theorem 2) we may assume that A is a nonzero indecomposable left ideal. If A is projective then the projective dimension is zero. So let A be not projective. Then by Lemma 3, $A = Rexe$ for some $0 \neq exe \in eNe$ (e being some primitive idempotent). We construct an infinite projective resolution of A

$$\cdots P_n \xrightarrow{f_n} P_{n-1} \cdots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \rightarrow 0$$

such that for every n, $\ker f_n \cong A_n \oplus B_n$ where A_n is nonprojective indecomposable left ideal of R and is of the form Rex_ne, $0 \neq ex_ne \in eNe$. Choose $P_0 = Re$ and let f_0 be the natural R-homomorphism of Re onto $Rexe$. Then $\ker f_0 = 1_R$ (exe) $= A_0 \oplus B_0$ where $A_0 = Rex_0e$ is not projective (sublemma). Suppose we have constructed $P_0, P_1, \cdots, P_n$ with exact sequence

$$0 \rightarrow \ker f_n \xrightarrow{\lambda} P_n \xrightarrow{f_n} P_{n-1} \rightarrow \cdots P_1 \rightarrow P_0 \xrightarrow{f_0} A$$

where λ is injection. By induction hypothesis $\ker f_n = A_n \oplus B_n$, where $0 \neq A_n = Rex_ne \subset Ne$.

Consider short exact sequences

$$0 \rightarrow 1_R(ex_ne) \xrightarrow{\sigma_n} Re \xrightarrow{\eta_n} A_n \rightarrow 0$$

and

$$0 \rightarrow D_{n+1} \xrightarrow{\sigma_n'} Q_n \xrightarrow{\eta_n'} B_n \rightarrow 0$$

where η_n is a natural R-homomorphism, σ_n is an injection and Q_n is some projective module.

Set $P_{n+1} = Q_n \oplus Re$ and $f_{n+1} = \lambda(\eta'_n \oplus \eta_n)$. Then $\ker f_{n+1} = \ker \eta'_n \oplus \ker \eta_n$. Also $\ker \eta_n = 1_R(ex_ne) = Rex_{n+1}e \oplus K$ (by sublemma). Thus f_{n+1} has the required property. Since P_{n+1} is projective, we have obtained the desired projective resolution of A.

Recall that if R is not a semisimple artinian ring then

$l.\ gl \dim R = 1 + \sup\{1.\dim_R A \mid A$ is a left ideal$\}$.

The previous theorem then yields the following

THEOREM 13. *Let R be a perfect left qp-ring. Then* $l.\ gl \dim R = 0, 1$, or ∞.

6. It is well known that a left hereditary semiprimary ring is also right hereditary [2]. Here we give an example of a local primary ring which is a left qp-ring but is not a right qp-ring.

EXAMPLE. Let F be a field which has an isomorphism $a \to \bar{a}$ that is not an automorphism, and let $\bar{F}$ be the subfield of the images $\bar{a}, a \in F$. Take x to be an indeterminate over F. Let $F[x]$ be the ring of polynomials of the form $a_0 + a_1x + a_2x^2, a_i \in F$; multiplication being defined by the rule $xa = \bar{a}x, x^3 = 0$ together with distributive law. It is well known that such rings are principal left ideal rings. Its radical $N = \{a_1x + a_2x^2 \mid a_i \in F\}$ is such that $N^2 \neq (0), N^3 = (0)$ and is a maximal left ideal of R. So R is a local perfect ring. Also N is not principal as a right ideal. So by Theorem 5, R is a left qp-ring but not a right qp-ring.

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