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COMPACT SUBSETS OF A TYCHONOFF SET

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The paper establishes a relation between the partial exponential law and the compactness of certain subsets of Tychonoff sets of multifunctions, and deduces consequences bearing on the Ascoli theorems established by Weston and Lin-Rose.

1. Introduction. The "Tychonoff set" is an abstraction of a class of sets arising in extensions of the classical Tychonoff theorem to multifunction context ([2], [5]). Extending the definition of the partial exponential law to multifunctions, we show that, when it is satisfied for a topology τ, certain subsets of a Tychonoff set are τ-compact. This approach — which is a non-trivial modification of the method introduced into function Ascoli theory by Noble [7] — will yield, in particular, sufficient conditions for compactness relative to the compact open topology.

In [6] Lin and Rose introduced a multifunction extension of the Kelley-Morse notion of even continuity, and proved a multifunction Ascoli theorem of the Weston type, without, however, showing that it contains the prototype [11, p. 20]. We deduce from our criterion a generalization of the Lin-Rose theorem. We show that this generalization contains the Weston Ascoli theorem and yields corollaries equivalent to the Tychonoff theorems for point-compact and point-closed multifunctions established in [2].

2. Multifunctions. We review the established definitions for multifunctions ([1], [9], [10]): Let X, Y be nonempty sets. A *multifunction* is a point to set correspondence $f\colon X \to Y$ such that, for all $x \in X, fx$ is a nonempty subset of Y. For $A \subseteq X, B \subseteq Y$ it is customary to write $f(A) = \bigcup_{x \in A} fx, f^-(B) = \{x : x \in X \text{ and } fx \cap B \neq \phi\}$ and $f^+(B) = \{x : x \in X \text{ and } fx \subseteq B\}$. If Y is a topological space, a multifunction $f\colon X \to Y$ is *point-compact* (*point-closed*) if fx is compact (closed) for all $x \in X$. If X, Y are topological spaces, a multifunction $f\colon X \to Y$ is *continuous* if $f^-(U), f^+(U)$ are open in X whenever U is open in Y. Henceforth the set of all continuous multifunctions (continuous functions) on a topological space X to a topological space Y will be denoted $\mathscr{C}(X, Y)(C(X, Y))$.

Let $\{Y_x\}_{x \in X}$ be a family of nonempty sets. The *m-product* $P\{Y_x : x \in X\}$ of the Y_x is the set of all multifunctions $f\colon X \to \bigcup_{x \in X} Y_x$

such that $fx \subseteq Y_x$ for all $x \in X$. In the case $Y_x = Y$ for all $x \in X$, the m-product of the Y_x, denoted Y^{mX}, is the set of all multifunctions on X to Y. For $x \in X$, the *x-projection* $pr_x: P\{Y_x: x \in X\} \to Y_x$ is the multifuntion defined by $pr_x f = fx$. If the Y_x are topological spaces, the *pointwise topology* τ_p on $P\{Y_x: x \in X\}$ is defined to be the topology having as open subbase the sets of the forms $pr_x^-(U_x), pr_x^+(U_x)$, where U_x is open in Y_x, $x \in X$ (**[5]**, **[8]**).

For $F \subseteq Y^{mX}, x \in X$, we write $F[x] = \bigcup_{f \in F} fx$. Let Y be a topological space. We say that a subset F of Y^{mX} is *pointwise bounded* if $F[x]$ has compact closure in Y for all $x \in X$. We say that a subset T of Y^{mX} is *Tychonoff* if, for every pointwise bounded subset F of T, $T \cap P\{\overline{F[x]}: x \in X\}$ is τ_p-compact. The following subsets of Y^{mX} are Tychonoff:

(1) Y^X, by the classical Tychonoff theorem.

(2) Y^{mX}, by the theorem of Lin [**5**, p. 400].

(3) The set of all point-closed members of Y^{mX}, by Corollary 2 of [**2**].

(4) The set of all point-compact members of Y^{mX}, by Corollary 3 of [**2**].

LEMMA 2.1. *If F is a pointwise bounded subset of a Tychonoff set T, then the τ_p-closure of F is compact.*

Proof. Let $\bar{F}$ denote the τ_p-closure of F. Since $P\{\overline{F[x]}: x \in X\} \cap T$ is a τ_p-compact subset of T, if suffices to show that $\bar{F} \subseteq P\{\overline{F[x]}: x \in X\}$. Let $f \in \bar{F}$. We must show that, for $x \in X, y \in fx$ and an open neighbourhood V of $y, F[x] \cap V \neq \emptyset$. Since $M = \{h: h \in T$ and $hx \cap V \neq \emptyset\}$ is a τ_p-neighborhood of f, there exists $h' \in M \cap F$. Then $h'x \cap V \neq \emptyset$ and $h'x \subseteq F[x]$, so $F[x] \cap V \neq \emptyset$.

Let X, Y be topological spaces. The multifunction $(f, x) \to fx$ on $Y^{mX} \times X$ to Y, or any restriction, will be denoted by the symbol ω. Let $F \subseteq Y^{mX}$. A topology τ on F is said to be *jointly continuous* if $\omega: (F, \tau) \times X \to Y$ is continuous [**8**, p. 48]. The *compact open topology* τ_c on Y^{mX} is defined to be the topology having as open subbase the sets of the forms $\{f: f(K) \subseteq U\}$, $\{f: fx \cap U \neq \emptyset$ for all $x \in K\}$, where K is a compact subset of X and U is open in Y ([**6**, p. 742], [**8**, p. 47]). Obviously τ_c is larger than τ_p.

3. Partial exponential law. Let X, Y, Z be topological spaces. An element $f \in Z^{m(X \times Y)}$ determines the function $\tilde{f}: x \to f(x, \cdot)$ on X to Z^{mY}. The function $\mu: f \to \tilde{f}$, called the *exponential map*, is a bijection of $Z^{m(X \times Y)}$ onto $(Z^{mY})^X$. It is clear that if $f \in \mathscr{C}(X \times Y, Z)$, then $\tilde{f}(x) = f(x, \cdot) \in \mathscr{C}(Y, Z)$ for all $x \in X$. When τ is a topology on

Z^{mY}, we say that (X, Y, Z, τ) *satisfies the partial exponential law* if $\mu(\mathscr{C}(X \times Y, Z)) \subseteq C(X, (\mathscr{C}(Y, Z), \tau))$.

We establish now the main theorem of the paper:

THEOREM 3.1. *Let T be a Tychonoff set of multifunctions on a topological space X to a topological space Y, and let τ be a topology on Y^{mX} such that (K, X, Y, τ) satisfies the partial exponential law for all compact spaces K. Then a subset F of T is τ-compact if*
(a) *F is τ-closed in T,*
(b) *F is pointwise bounded, and*
(c) *τ_p is jointly continuous on the τ_p-closure of F in T.*

Proof. Let $\bar{F}$ denote the τ_p-closure of F in T and let $\omega: (\bar{F}, \tau_p) \times X \to Y$. By (c), ω is continuous, so $\bar{F} \subseteq \mathscr{C}(X, Y)$. Since T is a Tychonoff set, (b) implies, by Lemma 2.1, that $\bar{F}$ is τ_p-compact. Then $\tilde{\omega}: (\bar{F}, \tau_p) \to (\mathscr{C}(X, Y), \tau)$ is continuous. Since $\tilde{\omega}$ is the inclusion map, $\bar{F} = \tilde{\omega}(\bar{F})$ is τ-compact. Since, by (a), F is τ-closed in $\bar{F}$, it follows that F is τ-compact.

The application of this theorem to τ_c depends on the following generalization to multifunctions of Lemma 1 of R. H. Fox [**3**, p. 430]:

LEMMA 3.2. *(X, Y, Z, τ_c) satisfies the partial exponential law.*

Proof. Let $f \in \mathscr{C}(X \times Y, Z)$. Let $x \in X$. Since $f(x, \cdot) = f \circ j$, where $j(y) = (x, y)$ $(y \in Y), f(x, \cdot)$ is continuous [**9**, p. 35]. Thus $\tilde{f}$ maps X into $\mathscr{C}(Y, Z)$. It remains to show that $\tilde{f}: X \to (\mathscr{C}(Y, Z), \tau_c)$ is continuous.

Let $M = \{h: h \in \mathscr{C}(Y, Z) \text{ and } h(K) \subseteq U\}$, where K is a compact subset of Y and U is open in Z. Let $x_0 \in \tilde{f}^{-1}(M)$. Then $f(x_0, \cdot) \in M$, so $\{x_0\} \times K \subseteq f^{+}(U)$. By the theorem of Wallace [**4**, p. 142], there is a neighbourhood V of x_0 such that $V \times K \subseteq f^{+}(U)$. Let $x \in V$. Then, for all $y \in K, \tilde{f}(x)y = f(x, y) \subseteq U$, so $\tilde{f}(x)(K) \subseteq U$. Thus $x \in \tilde{f}^{-1}(M)$, and we have shown that $\tilde{f}^{-1}(M)$ is open in X.

Let $M = \{h: h \in \mathscr{C}(Y, Z) \text{ and } hy \cap U \neq \emptyset \text{ for all } y \in K\}$, where K is a compact subset of Y and U is open in Z. Let $x_0 \in \tilde{f}^{-1}(M)$. Then $f(x_0, \cdot) \in M$, so $\{x_0\} \times K \subseteq f^{-}(U)$. There is a neighbourhood V of x_0 such that $V \times K \subseteq f^{-}(U)$. Let $x \in V$. Then, for all $y \in K, \tilde{f}(x)y \cap U \neq \emptyset$, so $\tilde{f}(x) \in M$, that is, $x \in \tilde{f}^{-1}(M)$, and we have shown that $\tilde{f}^{-1}(M)$ is open in X.

4. Even continuity. Let X, Y be topological spaces and let $F \subseteq Y^{mX}$. Following [**6**], we say that F is *evenly continuous* if, for each

$(x, y) \in X \times Y$ and each neighborhood V of y, there exist neighbourhoods U, W of x, y, respectively, such that

(a) $f \in F$ and $fx \cap W \neq \varnothing$ imply $U \subseteq f^{-}(V)$, and

(b) $f \in F, fx \cap W \neq \varnothing$ and $fx \subseteq V$ imply $f(U) \subseteq V$.

LEMMA 4.1. *Let X, Y be topological spaces and let $F \subseteq Y^{mX}$. If F is evenly continuous, τ_p on F is jointly continuous.*

Proof. Let $\omega: (F, \tau_P) \times X \to Y$. Suppose that $(f, x) \in \omega^{-}(V)$, where V is open in Y. Choose $y \in fx \cap V$. Then there exist open neighbourhoods U, W of x, y, respectively, such that $g \in F$ and $gx \cap W \neq \varnothing$ imply $U \subseteq g^{-}(V)$. Write $M = \{h: h \in F$ and $hx \cap W \neq \varnothing\}$. Then $M \times U$ is a neighbourhood of (f, x), which is contained in $\omega^{-}(V)$. Now suppose that $(f, x) \in \omega^{+}(V)$, where V is open in Y. Then $fx \subseteq V$. Choose $y \in fx$. There exist open neighbourhoods U, W of x, y, respectively, such that $g \in F, gx \cap W \neq \varnothing$ and $gx \subseteq V$ imply $g(U) \subseteq V$. Write $M = \{h: h \in F, hx \cap W \neq \varnothing$ and $hx \subseteq V\}$. Then $M \times U$ is a neighbourhood of (f, x), which is contained in $\omega^{+}(V)$.

COROLLARY 4.2. *Let X, Y be topological spaces and let $F \subseteq Y^{mX}$. If F is evenly continuous, then each member of F is continuous.*

The following result, which generalizes the Ascoli theorem of Lin and Rose [**6**, p. 746], contains also the Weston Ascoli theorem [**11**, p. 20]:

THEOREM 4.3. *Let T be a Tychonoff set of multifunctions on a topological space X to a topological space Y. Then a subset F of T is τ_c-compact if*

(a) *F is τ_c-closed,*

(b) *F is pointwise bounded, and*

(c) *F is evenly continuous.*

Proof. By Lemma 3.2, this theorem will follow as a corollary of Theorem 3.1 if we show that τ_p is jointly continuous on the τ_p-closure $\bar{F}$ of F. By (c) and Lemma 4.1, τ_p on F is jointly continuous. By (a), $F = \tilde{F}$, where $\tilde{F}$ is the τ_c-closure of F. Finally, by (c) and Lemma 3.1 of [**6**, p. 744], $\tilde{F} = \bar{F}$.

COROLLARY 4.4. *Let $(Y^{mX})_0((Y^{mX})_1)$ be the set of all point-compact (point-closed) multifunctions on a topological space X to a* *topological space Y. Then a subset F of* $(Y^{mX})_0((Y^{mX})_1)$ *is* τ_c*-compact if*

(a) *F is* τ_c*-closed,*
(b) *F is pointwise bounded, and*
(c) *F is evenly continuous.*

5. REMARKS. The Lin-Rose Ascoli theorem [**6**, p. 746], depends, apart from Lemma 3.1 of [**6**], on the Tychonoff theorem of Lin [**5**, p. 400]. Consequently, the Corollary 4.4 can be proved by the Lin-Rose argument, using the Tychonoff theorems of [**2**]. We will prove the converse implication: Let $\{Y_x\}_{x\in X}$ be a family of compact spaces. We will deduce from Corollary 4.4 that $F = (P\{Y_x : x \in X\})_0$ is τ_p-compact.

We may suppose the Y_x disjoint. Assign to X the discrete topology and let $Y = \bigcup_{x\in X} Y_x$ have the sum topology. We have $F \subseteq (Y^{mX})_0$ and, since X is discrete, F is evenly continuous [**6**, p. 743]. Since $F[x] = Y_x$ and Y_x is closed in Y, F is pointwise bounded. If we show that F is τ_p-closed, it will follow from Corollary 4.4 that F is τ_c-compact and therefore τ_p-compact. Let $\{f_\alpha\}$ be a net in F which is τ_p-convergent to an element $f \in (Y^{mX})_0$. Let $x \in X$. Let $y \in fx$, and let V be an open neighborhood of y. Since $\{h : h \in (Y^{mX})_0$ and $hx \cap V \neq \emptyset\}$ is a τ_p-neighborhood of $f, f_\alpha x \cap V \neq \emptyset$ eventually. Since $f_\alpha x \subseteq Y_x$, $Y_x \cap V \neq \emptyset$. This shows that $y \in \bar{Y}_x = Y_x$, proving that $f \in F$.

We prove similarly the same implication for $F = (P\{Y_x : x \in X\})_1$.

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