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ON TWO THEOREMS OF FROBENIUS

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This note contains simple proofs of two classical theorems of Frobenius, on nonnegative matrices. These concern powers of a primitive matrix and the maximal root of a principal submatrix of an irreducible matrix.

The purpose of this note is to give simple and straightforward proofs for two classical theorems of Frobenius **[1]**.

A matrix A is said to be *nonnegative* (*positive*) if all its entries are nonnegative (positive); we write $A \geqq 0$ $(A > 0)$. A nonnegative square matrix is called *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are square; otherwise A is *irreducible.*

It was shown by Frobenius **[1]** that a nonnegative square matrix has a real *maximal root* r such that $r \leqq |\lambda_t|$ for every root λ_t of A and that to r corresponds a nonnegative characteristic vector. Moreover, if A is irreducible, then the maximal root r of A is simple and there is a positive characteristic vector corresponding to it. An irreducible matrix is said to be *primitive* if its maximal root is *strictly* greater than the moduli of the other roots.

We prove the following remarkable two results due to Frobenius (see **[1]**; also Theorem 8 and Proposition 4, p. 69, in **[2]**).

THEOREM 1. *If A is primitive then*

$$A^m > 0$$

for some positive integer m.

THEOREM 2. *The maximal root of an irreducible matrix is greater than the maximal root of any of its principal submatrices.*

Proof of Theorem 1. Let A be a primitive matrix with maximal root r. Then the matrix $1/rA$ is primitive as well, its maximal root is 1, and all its other roots have moduli less than 1. Let

$$S^{-1}\left(\frac{1}{r}A\right)S = 1 \dotplus B, \tag{1}$$

where $1 \dot{+} B$ is, e.g., the Jordan normal form of $1/rA$. We can deduce immediately from (1) that:

(i) the moduli of all roots of B are less than 1 and therefore $\lim_{t\to\infty} B^t = 0$;

(ii) the first column of S is a character vector of A corresponding to the maximal root 1 and therefore has no zero coordinates;

(iii) the first row of S^{-1} is a characteristic vector of the transpose of $1/rA$ corresponding to its maximal root and thus cannot have zero coordinates.

Now,

$$\begin{aligned}\lim_{t\to\infty}\left(\frac{1}{r}A\right)^t &= \lim_{t\to\infty}(S(1\dot{+}B)S^{-1})^t \\ &= S(1\dot{+}(\lim_{t\to\infty}B^t))S^{-1} \\ &= S(1\dot{+}0)S^{-1}\end{aligned}$$

is a nonnegative matrix. But the (i, j) entry of $S(1\dot{+}0)S^{-1}$ is the nonzero product $S_{i1}(S^{-1})_{1j}$. Hence $S(1\dot{+}0)S^{-1}$ must be strictly positive, i.e.,

$$\lim_{t\to\infty}\left(\frac{1}{r}A\right)^t > 0.$$

It follows that for sufficiently large integer m,

$$\left(\frac{1}{r}A\right)^m > 0,$$

and therefore

$$A^m > 0.$$

In order to prove Theorem 2 we require the following lemma obtained in a more general form by Wielandt [3]:

Let A be an $n \times n$ irreducible matrix with maximal root r. If x is a nonnegative n-tuple, $x \neq 0$, and k a nonnegative number satisfying

$$Ax - kx \geqq 0,$$

then

$$k \leqq r. \tag{2}$$

Equality can hold in (2) *only if $x > 0$.*

Proof of Theorem 2. We can assume without loss of generality that the principal submatrix in question lies in the first t rows and first t columns, i.e., that

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix},$$

where B is the principal $t \times t$ submatrix. Let r and k be the maximal roots of A and B, respectively. Let y be a nonnegative characteristic vector of B corresponding to k and let

$$x = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

be the n-tuple whose first t coordinates are those of y and whose last $n - t$ coordinates are 0. Then

$$\begin{aligned} Ax &= \begin{bmatrix} By \\ Dy \end{bmatrix} \\ &= k\begin{bmatrix} y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Dy \end{bmatrix}, \\ Ax - kx &= \begin{bmatrix} 0 \\ Dy \end{bmatrix} \\ &\geqq 0. \end{aligned}$$

It follows immediately by the preceding lemma that

$$k < r.$$

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2. F. R. Gantmacher, *The theory of matrices*, Vol. 2, New York 1959.

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